Rezolvare subiecte Analiză matematică la Inginerie electrică restanțe toamnă 2025

Note Title 9/13/2025

II

1. a) Funcția $\sin z$, $z \in \mathbb{C}$

   b) $\operatorname{Im}(w) = ?$ $\quad w = \sin\left(\frac{5\pi}{3} - i\right)$

2. a) Seria Taylor a funcției exponențiale

   b) Seria Taylor pentru $f(x) = x^3 \cdot e^{-x^2}$, $x_0 = 0$.

3. a) Gradientul unui câmp scalar (definiție + 3 proprietăți)

   b) $\operatorname{grad} f = ?$ $\quad f(x,y,z) = x^3 y \ln(x^2 + z^2 + 1)$, $x, y, z \in \mathbb{R}$.

4. a) Funcția Beta (definiție + 3 proprietăți)

   b) $\int_0^1 x^3 \cdot \sqrt{x(1-x)}\, dx = ?$

1\. a) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}$.

b) $w = \sin\left(\frac{5\pi}{3} - i\right) = \dfrac{e^{i\left(\frac{5\pi}{3}-i\right)} - e^{-i\left(\frac{5\pi}{3}-i\right)}}{2i} = \frac{1}{2i}\left(e^{1+\frac{5\pi i}{3}} - e^{-1-\frac{5\pi i}{3}}\right)$

$= \frac{1}{2i}\left[e\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) - e^{-1}\left(\cos\left(-\frac{5\pi}{3}\right) + i\sin\left(-\frac{5\pi}{3}\right)\right)\right]$

$\cos\frac{5\pi}{3} = \cos\frac{6\pi-\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$\cos\left(-\frac{5\pi}{3}\right) = \cos\frac{5\pi}{3} = \frac{1}{2}$

$\sin\frac{5\pi}{3} = \sin\left(2\pi - \frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$\sin\left(-\frac{5\pi}{3}\right) = -\sin\frac{5\pi}{3} = \frac{\sqrt{3}}{2}$

$\Rightarrow w = \overset{-i}{\cancel{\frac{1}{2i}}}\left[e\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) - \frac{1}{e}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right] = \frac{-i}{2}\left(\frac{e}{2} - \frac{i\sqrt{3}e}{2} - \frac{1}{2e} - \frac{i\sqrt{3}}{2e}\right)$

$$w = \frac{-ie}{4} - \frac{\sqrt{3}\,e}{4} + \frac{i}{4e} - \frac{\sqrt{3}}{4e} \implies \operatorname{Im}(w) = -\frac{e}{4} + \frac{1}{4e}$$

2\. a) $$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{C}$$

b) $$f(x) = x^3 \cdot e^{-x^2} = x^3 \cdot \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{n!}, \quad x \in \mathbb{C}$$

$$= x^3 - x^5 + \frac{x^7}{2} - \dots$$

3. a)

$$\operatorname{grad} f = f'_x \vec{i} + f'_y \vec{j} + f'_z \vec{k}$$

$$\operatorname{grad}(cf) = c \operatorname{grad} f$$

$$\operatorname{grad}(f+g) = \operatorname{grad} f + \operatorname{grad} g$$

$$\operatorname{grad}(f \cdot g) = f \cdot \operatorname{grad} g + g \cdot \operatorname{grad} f$$

$$\operatorname{grad} |\vec{r}| = \operatorname{grad} \sqrt{x^2+y^2+z^2} = \frac{\vec{r}}{|\vec{r}|}$$

b)

$$f(x,y,z) = x^3 \cdot y \cdot \ln(x^2+z^2+1)$$

$$f'_x = 3x^2 y \ln(x^2+z^2+1) + x^3 y \cdot \frac{2x}{x^2+z^2+1}$$

$$f'_y = x^3 \ln(x^2+z^2+1)$$

$$f'_z = x^3 y \cdot \frac{2z}{x^2+z^2+1}$$

$$\Longrightarrow \operatorname{grad} f = \left(3x^2 y \ln(x^2+z^2+1) + \frac{2x^4 y}{x^2+z^2+1}\right)\vec{i} + x^3 \ln(x^2+z^2+1)\vec{j} + \frac{2x^3 yz}{x^2+z^2+1}\vec{k}$$

4. a) Funcția Beta $\beta(a,b) = \int_0^1 x^{a-1} \cdot (1-x)^{b-1} dx \quad , a,b>0$

$$\beta(a,b) = \beta(b,a) \quad , \quad a,b>0$$

$$\beta(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)} \quad , \quad a,b>0$$

$$\beta(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}} dx \quad , \quad a,b>0$$

b) $I = \int_0^1 x^3 \sqrt{x(1-x)}\, dx = \int_0^1 x^3 \cdot x^{\frac{1}{2}} (1-x)^{\frac{1}{2}} dx = \int_0^1 x^{3+\frac{1}{2}} (1-x)^{\frac{1}{2}} dx$

$a-1 = 3+\frac{1}{2}$ și $b-1 = \frac{1}{2} \implies a = 4+\frac{1}{2} = \frac{9}{2}$ și $b = \frac{3}{2}$

$$\implies I = B\left(\frac{9}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{9}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}+\frac{3}{2}\right)} = \frac{\frac{7\cdot5\cdot3}{16}\cdot\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{\Gamma(6)} = \frac{7\cdot5\cdot3\,\pi}{32\cdot5!} = \frac{7\pi}{256}$$

$$\Gamma\left(\frac{9}{2}\right) = \left(\frac{9}{2}-1\right)\Gamma\left(\frac{9}{2}-1\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{7}{2}\,\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$