Rezolvare subiecte Matematici speciale la ETTI restanțe toamnă 2025

Note Title 9/13/2025

I

1. Funcția Gamma (definiție + 3 proprietăți)

2. Convergență + calcul $\int_0^\infty \frac{x}{(x^2+2)(x^2+4)}\,dx$

3. $f(z) = \frac{e^{\pi z}}{z^2+4}$ a) care sunt punctele singulare ale lui f și cât sunt reziduurile lui f în aceste puncte

   b) $\int_{|z|=4} f(z)\,dz = ?$

4. $\iint_D y\cdot x^3\,dxdy$, $D = \text{int}\,\triangle ABC$ $A(1,0)$, $B(0,1)$, $C(0,0)$.

1. Funcția Gamma $\Gamma(a) = \int_0^\infty e^{-x} \cdot x^{a-1}\,dx \quad , \quad a > 0$

$$\Gamma(1) = 1$$

$$\Gamma(n) = (n-1)! \quad , \quad n \in \mathbb{N} \, , \, n \geq 2$$

$$\Gamma(a) = (a-1)\,\Gamma(a-1) \quad , \quad a > 1$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(a) \cdot \Gamma(1-a) = \frac{\pi}{\sin(\pi a)} \quad , \quad a \in (0,1)$$

2. Convergența $\int_0^\infty \frac{x}{(x^2+2)(x^2+4)}\,dx$

$$L = \lim_{x\to\infty} x^\alpha \cdot \frac{x}{(x^2+2)(x^2+4)} = \lim_{x\to\infty} \frac{\cancel{x^3}\cdot x}{\cancel{x^2}\left(1+\frac{2}{x^2}\right)\cancel{x^2}\left(1+\frac{4}{x^2}\right)} = 1$$

$$\alpha + 1 = 2+2$$

$$\alpha = 3$$

Pt că $\begin{array}{l} L = 1 > 0 \\ \alpha = 3 > 1 \end{array} \Rightarrow$ integrală convergentă

Pentru calcul notăm $x^2 = t \Rightarrow 2x\,dx = dt$

$x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$\Rightarrow \int_0^\infty \frac{x}{(x^2+2)(x^2+4)}\,dx = \frac{1}{2}\int_0^\infty \frac{2x\,dx}{(x^2+2)(x^2+4)} = \frac{1}{2}\int_0^\infty \frac{dt}{(t+2)(t+4)}$$

Descompunem în fracții simple $\frac{1}{(t+2)(t+4)} = \frac{A}{t+2} + \frac{B}{t+4}$

$$\Rightarrow \quad 1 = A(t+4) + B(t+2)$$

Pentru $t = -4 \Rightarrow 1 = B\cdot(-2) \Rightarrow B = -\frac{1}{2}$

$t = -2 \Rightarrow 1 = A\cdot 2 \Rightarrow A = \frac{1}{2}$

$$\Rightarrow \frac{1}{(t+2)(t+4)} = \frac{\frac{1}{2}}{t+2} - \frac{\frac{1}{2}}{t+4}$$

$$\Rightarrow I = \frac{1}{2}\int_0^\infty \left(\frac{\frac{1}{2}}{t+2} - \frac{\frac{1}{2}}{t+4}\right)dt = \frac{1}{2}\left(\frac{1}{2}\ln(t+2) - \frac{1}{2}\ln(t+4)\right)\Big|_0^\infty$$

$$= \frac{1}{4}\ln\frac{t+2}{t+4}\Big|_0^\infty = \lim_{t\to\infty}\frac{1}{4}\ln\frac{t+2}{t+4} - \frac{1}{4}\ln\frac{2}{4}$$

$$= \frac{1}{4}\ln\left(\lim_{t\to\infty}\frac{t\left(1+\frac{2}{t}\right)}{t\left(1+\frac{4}{t}\right)}\right) - \frac{1}{4}\ln\frac{1}{2} = \frac{1}{4}\ln 1 - \frac{1}{4}\ln\frac{1}{2} = \frac{1}{4}\ln\frac{1}{\frac{1}{2}} = \frac{1}{4}\ln 2$$

3) a) Punctele singulare ale funcției $f(z) = \dfrac{e^{\pi z}}{z^2+4}$ se obțin rezolvând ecuația $z^2+4=0$.

$$z^2 = -4 = -2^2 = i^2 \cdot 2^2 \Rightarrow z_{1,2} = \pm 2i \qquad \begin{matrix} z_1 = 2i \\ z_2 = -2i \end{matrix} \text{ poli de ordin 1}$$

$$z^2+4 = (z-z_1)(z-z_2)$$

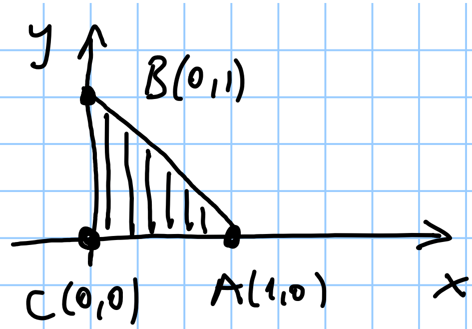
$$\operatorname{Rez}(f, z_1) = \lim_{z \to z_1} (z-z_1) f(z) = \lim_{z \to z_1} \cancel{(z-z_1)} \frac{e^{\pi z}}{\cancel{(z-z_1)}(z-z_2)} = \frac{e^{\pi z_1}}{z_1 - z_2}$$

$$= \frac{e^{\pi \cdot 2i}}{2i+2i} = \frac{1}{4i}$$

$$\operatorname{Rez}(f, z_2) = \lim_{z \to z_2} (z-z_2) f(z) = \frac{e^{\pi z_2}}{z_2 - z_1} = \frac{e^{-2\pi i}}{-2i-2i} = -\frac{1}{4i}$$

b) $$\int_{|z|=4} f(z)\,dz = 2\pi i \left[\operatorname{Rez}(f, z_1) + \operatorname{Rez}(f, z_2)\right] = 2\pi i \left(\frac{1}{4i} - \frac{1}{4i}\right) = 0.$$

4.



$$D = \{(x,y) \in \mathbb{R}^2,\quad x \in [0,1],\ y_{AC} \le y \le y_{AB}\}$$

$$AC:\ \frac{x - x_C}{x_A - x_C} = \frac{y - y_C}{y_A - y_C} \iff \frac{x-0}{1-0} = \frac{y-0}{0-0}$$

$$\iff \frac{x}{1} = \frac{y}{0} \iff x \cdot 0 = y \cdot 1 \iff \underline{y = 0}$$

$$\implies y_{AC} = 0$$

$$AB:\ \frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} \iff \frac{x-1}{0-1} = \frac{y-0}{1-0} \iff \frac{x-1}{-1} = \frac{y}{1} \iff -y = x-1$$

$$\implies y = -(x-1) = -x+1 \implies y_{AB} = 1-x$$

$$\implies \iint_D y \cdot x^3\,dx\,dy = \int_0^1 \left(\int_0^{1-x} y\,x^3\,dy\right) dx = \int_0^1 x^3 \left(\int_0^{1-x} y\,dy\right) dx = \int_0^1 x^3\, \frac{y^2}{2}\Big|_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x^3(1-x)^2\,dx = \frac{1}{2}\int_0^1 (x^3 - 2x^4 + x^5)\,dx = \frac{1}{2}\left(\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = \frac{1}{120}$$