

The Design of Feedback Control Systems

Paula Raica

Department of Automation

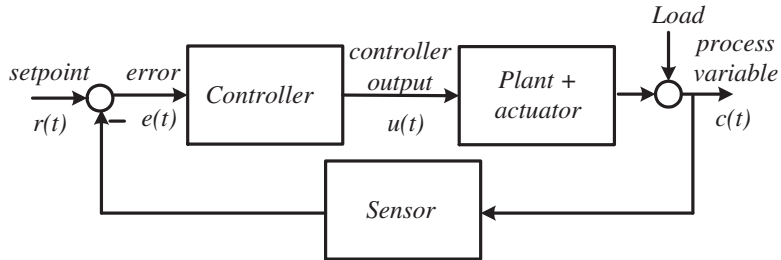
71 Dorobantilor Str., room C21, tel: 0264 - 401267

26-28 Baritiu Str., room C14, tel: 0264 - 202368

email: Paula.Raica@aut.utcluj.ro

Technical University of Cluj-Napoca

Control system



- The performance of a feedback control system: stability, acceptable response, less sensitive to system parameter changes, a minimum steady-state error for input commands and is able to reduce the effect of undesirable disturbances.
- The design of a control system = the arrangement of the system structure and the selection of suitable components and parameters.
- To alter the system response, an additional component is inserted within the structure of the feedback system: *compensator*.

Approaches to System Design

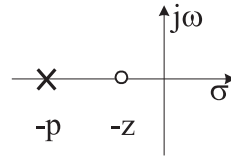
- Specifications a control system:
 - In time-domain: (ex) peak time, a maximum overshoot and settling time for a step input $\Rightarrow$ the desirable location of the poles and zeros of the closed-loop transfer function
 - In frequency domain: resonant frequency, the bandwidth, the phase margin.
- Assume that the process has been improved as much as possible and the transfer function representing the process is unalterable.
- A compensator must be a physical device. It may be an may be electronic device, an RC network (electrical, mechanical, hydraulic, pneumatic or combinations), or a computer/signal processor.

Root-locus approach

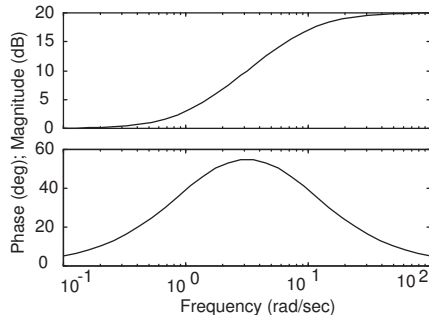
- RL: a graphical method for determining the locations of all closed-loop poles as some parameter (usually the gain) is varied from zero to infinity.
- LR yields a clear indication of the effects of parameter adjustment and information on the transient response.
- For control system design purposes it is often necessary to reshape the RL to meet the performance specifications.
- In the design via the RL method: the root loci of the system are reshaped through the use of a compensator so that a pair of dominant closed-loop poles can be placed at the desired location.

Phase-lead network

$$G_c(s) = \frac{k(s + z)}{s + p}, \quad |z| < |p|$$



The frequency response is:



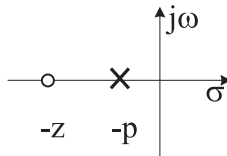
May be used to alter the transient state of the closed-loop system

Example (lead element)

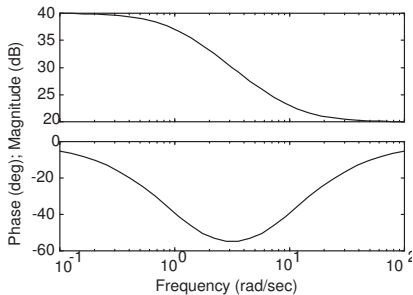
$$G(s) = \frac{s + 1}{s + 10}$$

Phase-lag network

$$G_c(s) = \frac{k(s + z)}{s + p}, \quad |z| > |p|$$



The frequency response:



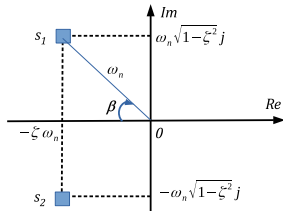
May be used to alter the steady-state error of the closed-loop system

Example (lag element)

$$G(s) = \frac{s + 10}{s + 1}$$

Remember: poles and transient response characteristics

- The poles of a system can be related to the characteristics of the transient



response

- Ex:

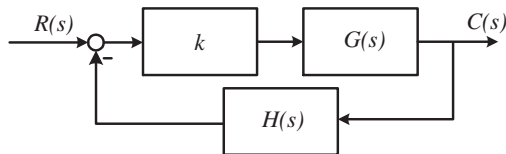
$$t_s = \frac{4}{\zeta \omega_n}$$

$$M_P = e^{-\pi \zeta / \sqrt{1-\zeta^2}}, \quad \cos \beta = \zeta$$

$$t_r = \frac{\pi - \beta}{\omega_d}, \quad t_p = \frac{\pi}{\omega_d}$$

$$(\omega_d = \omega_n \sqrt{1 - \zeta^2})$$

Remember: root locus



- Open-loop transfer function: $H_d(s) = kG(s)H(s)$
- Closed-loop transfer function: $\frac{C(s)}{R(s)} = \frac{kG(s)}{1 + kG(s)H(s)}$
- The root locus: a plot of the **closed-loop poles** for $k \in [0, \infty)$, i.e. the roots of: $1 + kG(s)H(s) = 0$.
- **Angle condition:** $\angle kG(s)H(s)|_{s \in LR} = -180^\circ$
- **Magnitude condition:** $|kG(s)H(s)|_{s \in LR} = 1$

Phase-lead comp. based on RL

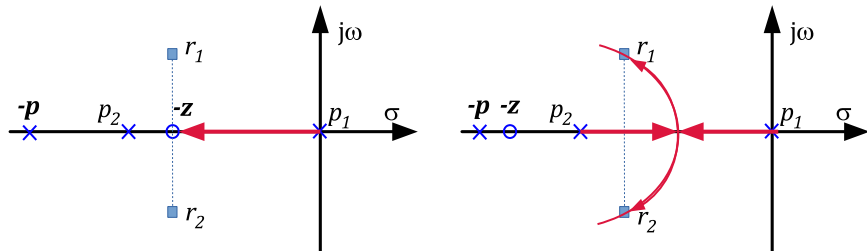
- Time-domain specifications:
 - location of closed-loop poles,
 - overshoot,
 - settling time,
 - damping factor
 - etc.
- Consider a design problem in which the original system either is unstable for all values of gain or it is stable but has undesirable transient response characteristics
- Insert an appropriate lead compensator in cascade with the feed-forward transfer function.

Phase-lead comp. based on RL

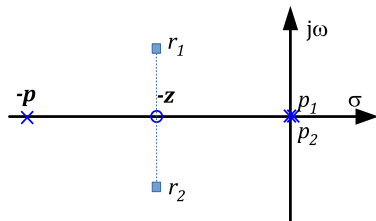
- 1 Closed-loop specifications $\rightarrow$ desired location for the dominant roots ($r_{1,2}$).
- 2 Sketch the uncompensated root locus and determine whether the desired root locations can be realized with an uncompensated system
- 3 Select a compensator $G_c(s) = \frac{k(s+z)}{s+p}$, $p > z > 0$.
- 4 Place in the complex plane all the known poles and zeros of the open-loop system.
- 5 Place the zero of the phase-lead network directly below the desired root location (or to the left of the first two real poles)
- 6 Determine the pole location so that the total angle at the desired root location is -180° (the angle condition is fulfilled for $r_{1,2}$, i.e. the desired poles are on the RL of the compensated system).
- 7 Evaluate the total system gain from the magnitude condition for $s = r_{1,2}$.

Phase-lead compensation

Add the compensator zero to the left of the first two real poles (so $r_{1,2}$ are the dominant poles).



If the compensator zero is between (p_1 and p_2), the RL branch (red) is closer to the origin than $r_{1,2}$, $\Rightarrow$ the pole on this branch will be dominant.



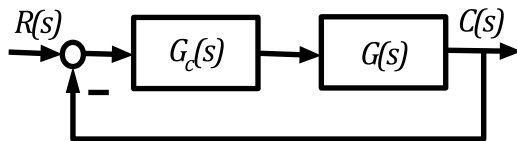
Phase-lead compensation

- If $G(s)$ is the process transfer function (known) and the compensator is $G_c(s) = \frac{k(s+z)}{s+p}$: k and $p > z > 0$ - unknown
 - Choose the zero of the compensator $-z$.
 - Compute the pole of the compensator from the angle condition
 - Compute the gain k from the magnitude condition
- If $G(s)$ is the process transfer function (known) and the compensator is partially known (k and z or p - unknown):
 - Compute the pole **or** the zero of the compensator from the angle condition
 - Compute the gain k from the magnitude condition.

Example 1

Let us consider a single-loop feedback control system with the open-loop (process) transfer function:

$$G(s) = \frac{1}{s^2}$$



We will design a closed-loop control system with the specifications:

- Settling time, $t_s \leq 4$ seconds
- Percent overshoot for a step input $M_p \leq 35\%$.

The open-loop system $G(s)$ is unstable as it has two poles in 0.

Example 1

- Consider a simple P compensator $G_c(s) = k$ and try to obtain the value of the gain k so that the design specifications are met.
- The closed-loop transfer function with the P-compensator is:

$$T(s) = \frac{kG(s)}{1 + kG(s)} = \frac{k \frac{1}{s^2}}{1 + k \frac{1}{s^2}} = \frac{k}{s^2 + k}$$

⇒ undamped oscillations.

- Select a lead compensator $G_c(s)$, where

$$G_c(s) = \frac{k(s + z)}{s + p}, \quad p > z > 0$$

- Use the desired specifications for the closed-loop system to obtain a pair of dominant poles.

Example 1

The admissible area for the dominant closed-loop poles:

- The damping ratio of the compensated system ζ :

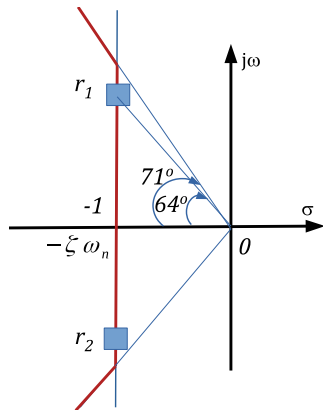
$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 0.35 \Rightarrow \zeta \geq 0.325$$

$$\Rightarrow \beta \leq \arccos \zeta = 71^\circ$$

- The settling time requirement is

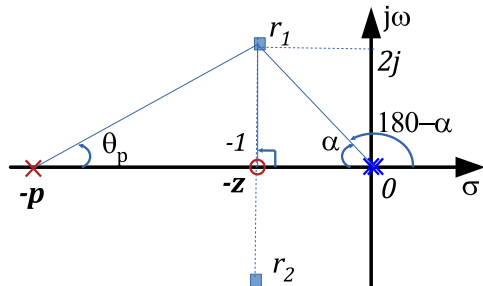
$$t_s = \frac{4}{\zeta\omega_n} \leq 4 \Rightarrow \zeta\omega_n \geq 1 \Rightarrow -\zeta\omega_n \leq -1$$

- All complex poles located inside an angle of 71° and to the left of the vertical line at -1 will fulfill the specifications.
- $r_{1,2} = -1 \pm 2j$, where the angle is $\alpha \approx 64^\circ$.



Example 1

- Place all the (known) open-loop poles and zeros in the complex plane. Place the dominant poles $r_{1,2}$.



- Place the compensator zero below r_1 (it is on the left of two open-loop poles $(0, 0)$): $-z = -1$.
- Place the compensator pole on the negative real axis and determine its value so that the dominant poles $r_{1,2}$ are on the RL of the compensated system, i.e. the angle condition is satisfied for $r_{1,2}$:

Example 1

- Angle condition: $\angle G_c(s)G(s)|_{s=r_1} = -180^\circ$.

$$\angle \frac{k(s+z)}{s+p} \cdot \frac{1}{s^2} \bigg|_{s=r_1} = -180^\circ$$

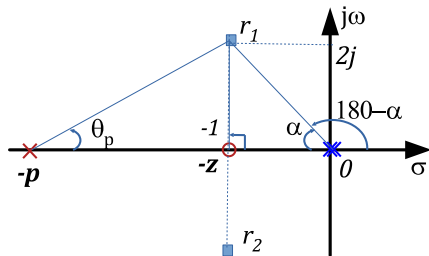
$$\angle G_c(s)G(s)|_{s=r_1} = \angle k + \angle(r_1 + z) - \angle(r_1 + p) - 2\angle(r_1) = -180^\circ$$

- From the figure: $\angle k = 0$, $\angle(r_1 + z) = \angle(r_1 + 1) = 90^\circ$, $\angle(r_1 + p) = \theta_p$, $\angle(r_1) = 180^\circ - \alpha$, $\alpha = 64^\circ$.

$$\angle G_c(s)G(s)|_{s=r_1} = 90^\circ - \theta_p - 2(180^\circ - 64^\circ) = -142^\circ - \theta_p = -180^\circ$$

$$\Rightarrow \theta_p = 38^\circ.$$

Example 1



- In the right triangle $(r_1, -z, -p)$:

$$\tan \theta_p = \tan 38^\circ = \frac{2}{p-1} \Rightarrow p-1 = \frac{2}{\tan 38^\circ} \Rightarrow p = 3.55$$

- The compensator is now: $G_c(s) = \frac{k(s+1)}{s+3.55}$.
- If $k \in [0, \infty)$ the RL of the closed-loop system will pass through $r_{1,2}$.

Example 1

- The open-loop transfer function of the compensated system is:

$$G_c(s)G(s) = \frac{k(s+1)}{s^2(s+3.55)}$$

- The gain k is evaluated from the magnitude condition:

$$|G_c(s)G(s)|_{s=r_1} = 1 \quad \Rightarrow \quad \left| \frac{k(s+1)}{s^2(s+3.55)} \right|_{s=-1+2j} = 1$$

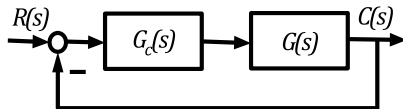
$$k = \left| \frac{s^2(s+3.55)}{s+1} \right|_{s=-1+2j} = \left| \frac{(-1+2j)^2(-1+2j+3.55)}{-1+2j+1} \right|$$

$$k = \frac{(\sqrt{5})^2 \cdot \sqrt{2.55^2 + 2^2}}{2} = 8.1$$

$$G_c(s) = \frac{8.1(s+1)}{s+3.55}$$

Example PI

- Consider a closed-loop control system with a plant $G(s) = \frac{1}{(s + \frac{1}{2})(s + 4)}$.
- Determine a PI controller $G_c(s) = K_P + \frac{K_I}{s}$, so that the dominant closed-loop poles are $r_{1,2} = -0.5 \pm j$.

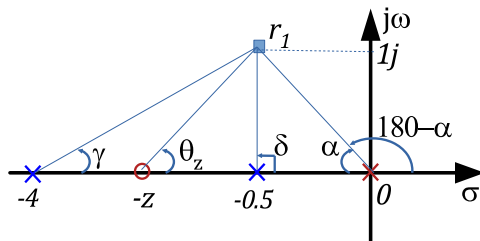


- The controller can be written (to emphasize the poles and zeros):

$$G_c(s) = K_P + \frac{K_I}{s} = K_P \frac{s + K_I/K_P}{s} = k \frac{s + z}{s}$$

- The compensator zero (z) will be calculated from the angle condition and the gain (k) from the magnitude condition.

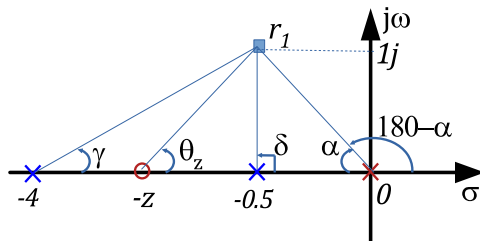
Example PI



- Place the poles that are known: the poles of the process (-4 and -0.5) and the pole of the compensator (0).
- The zero will be *calculated* from the angle condition:

$$\angle G_c(s)G(s)|_{s=r_1} = -180^\circ \Rightarrow \angle \frac{k(s+z)}{s(s+\frac{1}{2})(s+4)} \bigg|_{s=r_1} = -180^\circ$$

Example PI



$$\angle k + \angle(r_1 + z) - \angle r_1 - \angle(r_1 + \frac{1}{2}) - \angle(r_1 + 4) = -180^\circ$$

$$0 + \theta_z - (180^\circ - \alpha) - \delta - \gamma = -180^\circ$$

$$\theta_z - (180^\circ - \underbrace{\arctan \frac{1}{0.5}}_{63.4^\circ}) - 90^\circ - \underbrace{\arctan \frac{1}{4 - 0.5}}_{15.9^\circ} = -180^\circ$$

$$\theta_z = 42.5^\circ \Rightarrow \Delta(r_1, -0.5, -z): \tan \theta_z = \tan 42.5 = \frac{1}{z - 0.5} \Rightarrow z = 1.59$$

Example PI

- The compensator is now: $G_c(s) = \frac{k(s + 1.59)}{s}$.
- The gain k is calculated from the magnitude condition:

$$|G_c(s)G(s)|_{s=r_1} = \left| \frac{k(s + 1.59)}{s(s + \frac{1}{2})(s + 4)} \right|_{s=-0.5+j} = 1$$

$$k = \left| \frac{(-0.5 + j)(-0.5 + j + \frac{1}{2})(-0.5 + j + 4)}{(-0.5 + j + 1.59)} \right|$$

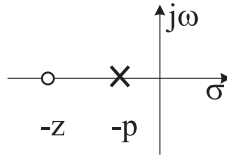
$$k = \frac{\sqrt{0.5^2 + 1^2} \cdot 1 \cdot \sqrt{3.5^2 + 1^2}}{\sqrt{1.09^2 + 1^2}} = 2.75$$

- The compensator: $G_c(s) = 2.75 \frac{s + 1.59}{s} = 2.75 + \frac{4.37}{s}$

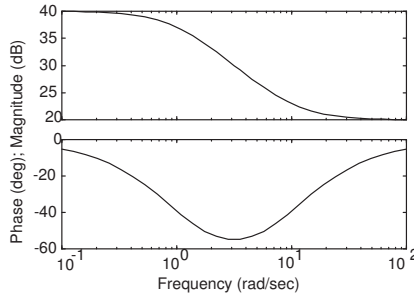
Phase-lag compensation using root locus

Phase-lag network

$$G_c(s) = \frac{k(s + z)}{s + p}, \quad |z| > |p|$$



The frequency response:



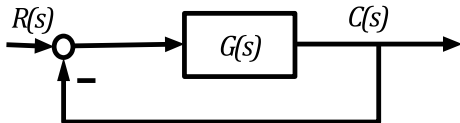
May be used to alter the steady-state error of the closed-loop system

Example (lag element)

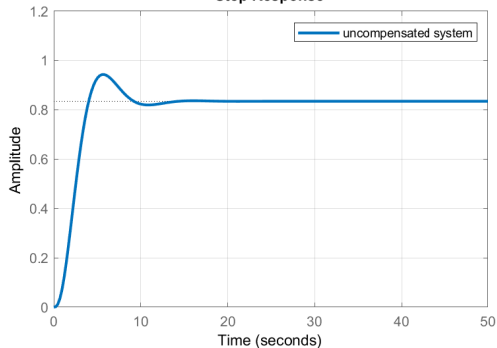
$$G_c(s) = \frac{s + 10}{s + 1}$$

Compensating the steady-state error

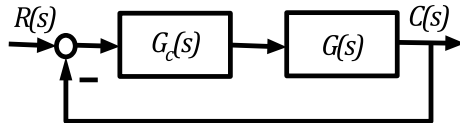
Uncompensated system



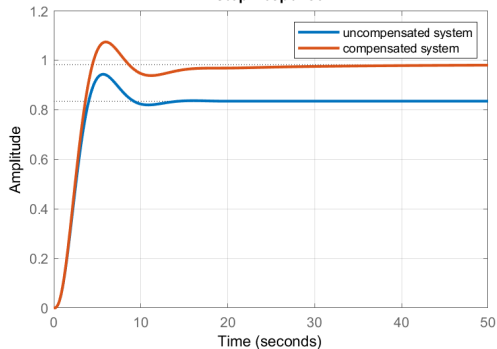
Step Response



Compensated system

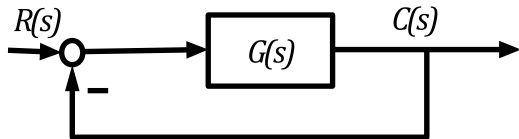


Step Response



Steady-state error

- (see Chapter 3, section 3.5.3 on static error constants)
- Consider a closed-loop negative feedback control system



$$G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{s^N \prod_{i=1}^Q (s + p_i)}$$

- $G(s)$ = the open-loop transfer function
- unity feedback
- N = type number
- The steady-state error:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} (r(t) - c(t))$$

$$e_{ss} = \lim_{s \rightarrow 0} s(R(s) - C(s)) = \lim_{s \rightarrow 0} sR(s) \left(1 - \frac{G(s)}{1 + G(s)}\right) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

Steady-state error

Step input: $r(t) = 1 \Rightarrow R(s) = \frac{1}{s}$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s} \frac{1}{1 + G(s)} = \frac{1}{1 + G(0)} = \frac{1}{1 + K_p}$$

$$K_p = \lim_{s \rightarrow 0} G(s) = \text{the position error constant}$$

■ $N = 0$:

$$G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{\prod_{i=1}^Q (s + p_i)} \Rightarrow K_p = \text{const} \Rightarrow e_{ss} = \text{const}$$

■ $N \geq 1$:

$$G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{s^N \prod_{i=1}^Q (s + p_i)} \Rightarrow K_p = \infty \Rightarrow e_{ss} = 0$$

Steady-state error

Ramp input: $r(t) = t \Rightarrow R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s^2} \frac{1}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{s(1 + G(s))} = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{K_v}$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = \text{the velocity error constant}$$

■ $N = 0$: $G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{\prod_{i=1}^Q (s + p_i)} \Rightarrow K_v = 0 \Rightarrow e_{ss} = \infty$

■ $N = 1$:

$$G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{s \prod_{i=1}^Q (s + p_i)} \Rightarrow K_v = \text{const} \Rightarrow e_{ss} = \text{const}$$

■ $N \geq 2$:

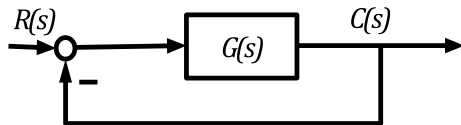
$$G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{s^N \prod_{i=1}^Q (s + p_i)} \Rightarrow K_v = \infty \Rightarrow e_{ss} = 0$$

Phase-lag comp. based on RL

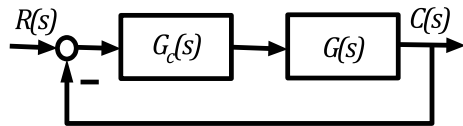
- Consider the problem of finding a suitable compensation network for the case where the system exhibits satisfactory transient-response characteristics but unsatisfactory steady-state characteristics
- Compensation in this case essentially consists of increasing the open-loop gain without appreciably changing the transient response characteristics.
- To avoid an appreciable change in the root loci the angle contribution of the lag network should be limited to a small amount, say 5° ;
- To assure this, we place the pole and zero of the lag network relatively close together and near the origin of the s-plane.

Compensating the steady-state error

Uncompensated system (before lag compensation)



Compensated system (after lag compensation)



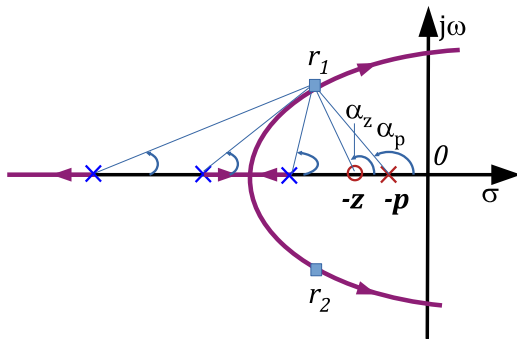
- Add a lag compensator $G_c(s) = \frac{s+z}{s+p}$, where $z > p > 0$
- $r_{1,2}$ are the dominant poles ($\Rightarrow$ the transient response) of the closed-loop system before lag compensation.
- The lag compensator should not change the transient response (keep the dominant poles in approximately the same position)
- $r_{1,2}$ satisfy the angle condition for the uncompensated RL:

$$\angle G(s)|_{s=r_1} = -180^\circ$$

Phase-lag comp. based on RL

- The angle condition for the compensated RL:

$$\angle G(s)G_c(s)|_{s=r_1^*} = \angle G(s) \frac{s+z}{s+p} \Big|_{s=r_1^*} = \angle G(s) + \underbrace{\angle s+z - \angle s+p}_{< 5^\circ} \Big|_{s=r_1^*} = -180^\circ$$



- $-z$ and $-p$ should be very close to each other to keep the dominant poles in almost the same location.

Phase-lag comp. based on RL

Consider a type 0 system for a step input, or a type 1 system for a ramp input.

- For the uncompensated system:

$$K_p = \lim_{s \rightarrow 0} G(s), \quad e_{ss} = \frac{1}{1 + K_p}$$

$$K_v = \lim_{s \rightarrow 0} sG(s), \quad e_{ss} = \frac{1}{K_v}$$

- For the compensated system:

$$K_{pcomp} = \lim_{s \rightarrow 0} G(s) \frac{s+z}{s+p} = \frac{z}{p} K_p, \quad e_{ss} = \frac{1}{1 + K_{pcomp}}$$

$$K_{vcomp} = \lim_{s \rightarrow 0} sG(s) \frac{s+z}{s+p} = \frac{z}{p} K_v, \quad e_{ss} = \frac{1}{K_{vcomp}}$$

- The ratio $\frac{z}{p}$ should be large to improve the steady-state error.

Phase-lag comp. based on RL

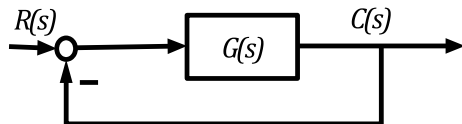
- The zero ($-z$) and the pole ($-p$) of the compensator should be close to each other to maintain location of the dominant closed-loop poles (and the transient response characteristics)
- The zero ($-z$) and the pole ($-p$) of the compensator should have a large ratio $\frac{z}{p}$ to improve the steady-state error.
- Solution: place them close to the origin *comparing to the location of the dominant poles*:
 - Choose the zero of the compensator small comparing to the natural frequency of the dominant poles: $z \approx \frac{\omega_n}{10}$
 - Compute the pole from the ratio $\frac{z}{p}$.

The algorithm

- 1 Determine the transient performance specifications for the system and locate suitable dominant root locations on the uncompensated root locus that will satisfy the specifications
- 2 Determine the dominant poles of the uncompensated system $r_{1,2}$
- 3 Calculate the system error constant at the dominant root location for the uncompensated system (before lag compensation): K_p or K_v .
- 4 Calculate the ratio z/p from the desired error constant and the uncompensated error constant.
- 5 Locate the zero close to the origin with respect to ω_n of dominant poles (for example $z \approx \frac{\omega_n}{10}$)
- 6 Determine the pole $-p$ from the ratio z/p .

Example

- Consider an closed-loop system with a **ramp** input ($r(t) = t, t \geq 0$) and the open-loop transfer function:



$$G(s) = \frac{5}{s(s+2)}$$

Design a lag compensator so that the velocity error constant is $K_{vcomp} = 20$ (or the steady-state error for a ramp input is $e_{ss} = \frac{1}{20} = 0.05$).

- Before the lag compensation (for the uncompensated system), the closed-loop transfer function and the closed-loop poles:

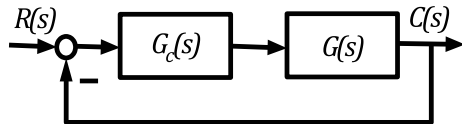
$$G_0(s) = \frac{G(s)}{1 + G(s)} = \frac{5}{s^2 + 2s + 5}, \quad r_{1,2} = -1 \pm 2j$$

- The static velocity error constant is:

$$K_v = \lim_{s \rightarrow 0} s \frac{5}{s(s+2)} = \frac{5}{2} = 2.5 \quad (e_{ss} = \frac{1}{2.5} = 0.4)$$

Example

- Add a lag compensator:



$$G_c(s) = \frac{s + z}{s + p} \quad \text{with } z > p > 0$$

- The velocity error constant for the compensated system:

$$K_{vcomp} = \lim_{s \rightarrow 0} s \frac{5}{s(s+2)} \frac{s+z}{s+p} = 2.5 \frac{z}{p} = 20$$

- The ratio $\frac{z}{p} = \frac{20}{2.5} = 8$.
- The natural frequency of the dominant poles $r_{1,2}$: $\omega_n = \sqrt{1^2 + 2^2} = \sqrt{5}$.
- Choose the zero so that $z < \omega_n/10$, for example: $z = 0.1$ (the zero at $-z = -0.1$) and the pole is:

$$\frac{z}{p} = 8 \Rightarrow p = \frac{0.1}{8} = 0.0125 \Rightarrow -p = -0.0125 \Rightarrow G_c(s) = \frac{s + 0.1}{s + 0.0125}$$

Example

The difference of the angles from $-p$ and $-z$ at the desired root is approximately 1° , and therefore $s = -1 \pm j2$ is still the location of the dominant roots.

