

Sampled-data systems. Digital control systems

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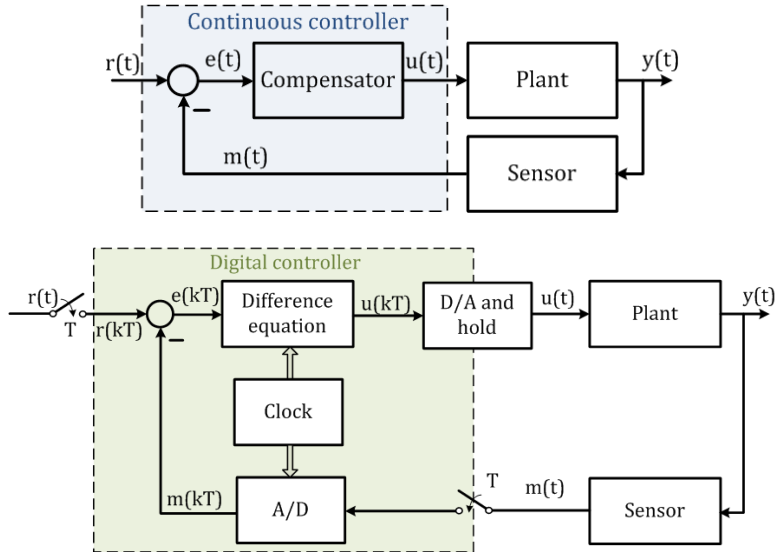
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Continuous control vs. digital control



Introduction

- The analog-to-digital (A/D) converter: samples a physical variable $m(t)$ and converts it into a binary number.
- Conversion of the analog signal $m(t)$ to $m(kT)$, occurs repeatedly at instants of time T seconds apart.
 - T is the sample period
 - $1/T$ is the sample rate in Hertz.
- The compensator algorithm is usually in the form of difference equations $\Rightarrow$ a discrete $u(kT)$ at each sample instant.
- $u(kT)$ is converted to a continuous $u(t)$ by the digital-to-analog (D/A) converter and the hold:
 - the D/A converter changes the binary number to an analog voltage,
 - a zero-order hold maintains that same voltage throughout the sample period.
- The resulting $u(t)$ is then applied to the actuator in the same manner as the continuous implementation.

The advantages of using digital control:

- Improved measurement sensitivity; the use of digitally coded signals
- Reduced sensitivity to signal noise
- The capability to easily reconfigure the control algorithm in software
- Wide application of digital devices and communications
- Low-energy signals required by digital sensors and devices

Two basic techniques for finding the difference equations for the digital controller:

- Discrete equivalent - consists of designing a continuous compensation using methods described in the previous chapters, then approximating the compensator transfer function using methods that will be described in the following sections.

$$G_c(s) \rightarrow G_c(z) \rightarrow \text{difference equation}$$

- Discrete design - the difference equations are found directly without designing the continuous compensator first.

Shannon's sampling theorem

A function $f(t)$ that has a bandwidth ω_b is uniquely determined by a discrete set of sample values provided that the sampling frequency is greater than $\omega_s = 2\omega_b$.

The sampling frequency $\omega_s = 2\omega_b$ is called the *Nyquist frequency*

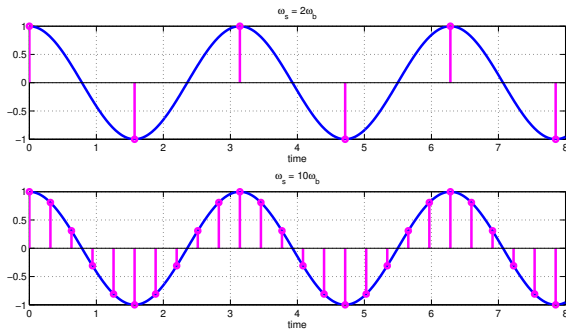
- A useful rule in applications is to sample the signal at about 10 times higher than the highest frequency thought to be present.
- Sampling period (or sampling time) is determined from ω_s (*in rad/s*):

$$\omega_s = 2\pi f_s = \frac{2\pi}{T_s} \quad \Rightarrow \quad T_s = \frac{2\pi}{\omega_s}$$

Sampling. Example 1

$u(t) = \cos 2t$, the frequency is: $\omega_b = 2 \text{ rad/s}$.

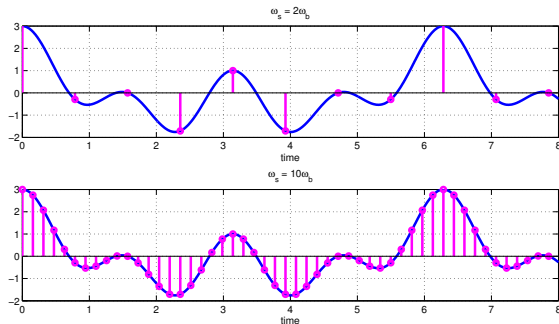
- $\omega_s = 2\omega_b = 4 \text{ rad/s}$, $T = 2\pi/\omega_s = 1.57\text{s}$.
- $\omega_s = 10\omega_b = 20 \text{ rad/s}$, $T = 2\pi/\omega_s = 0.314\text{s}$.



Sampling. Example 2

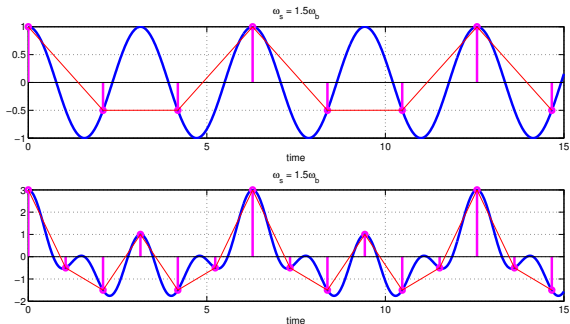
$u(t) = \cos t + \cos 2t + \cos 4t$ the highest frequency is: $\omega_b = 4$ rad/s.

- $\omega_s = 2\omega_b = 8$ rad/s, $T = 2\pi/\omega_s = 0.785$ s.
- $\omega_s = 10\omega_b = 40$ rad/s, $T = 2\pi/\omega_s = 0.157$ s.



Sampling. Aliasing

- If a signal is sampled below Shannon's limit, then a lower frequency signal, called an *alias* may be constructed.
- Example: $u(t) = \cos(2t)$, and $u(t) = \cos t + \cos 2t + \cos 4t$, the sampling frequency: $\omega_s = 1.5\omega_b$.
- Apparently the sampled signal has a different form, with a lower frequency. This situation is known as *aliasing*.



Sampling in control systems

- Several approaches are available.
- It is rather a problem of experience than some exact procedure.
- Shannon's theorem usually gives a lower limit for the sampling frequency.
- Choose the sampling frequency as:

$$6\omega_b < \omega_s < 25\omega_b$$

(two to nine samples on the rise time of a step response)

- For example, for a first order system $G_1(s) = \frac{1}{T_0s + 1}$, the sampling period may be chosen as:

$$\frac{T_0}{4} < T_s < T_0$$

Sampling. Example

$$G_1(s) = \frac{1}{T_0 s + 1}, \quad G_2(s) = \frac{1}{\frac{1}{\omega_n^2} s^2 + \frac{2\zeta}{\omega_n} s + 1}$$

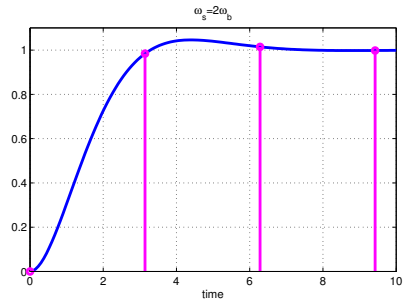
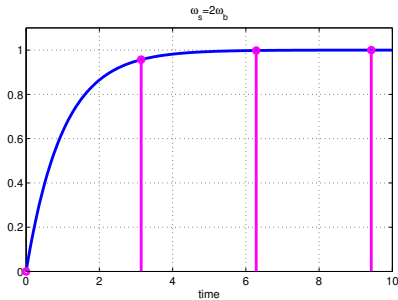
- For $T_0 = 1$, $\zeta = 0.7$ and $\omega_n = 1$ the systems' bandwidth frequency may be approximated by $\omega_{b1} = 1/T_0$ and $\omega_{b2} = \omega_n$.
- Shannon theorem:

$$\omega_{s1} = 2\omega_{b1} = \frac{2\pi}{T_s}, \quad T_s = \frac{2\pi}{2\omega_{b1}} = \frac{2\pi T_0}{2} = \pi T_0$$

$$\omega_{s2} = 2\omega_{b2} = \frac{2\pi}{T_s}, \quad T_s = \frac{2\pi}{2\omega_{b2}} = \frac{2\pi}{2\omega_n} = \frac{\pi}{\omega_n}$$

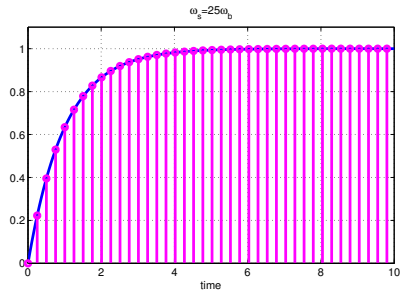
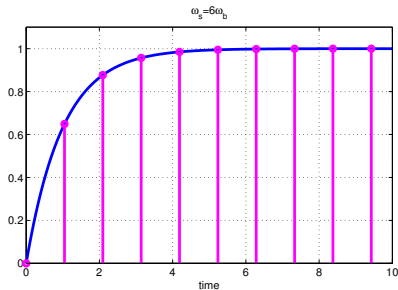
Sampling. Example

Shannon theorem: $\omega_s = 2\omega_b$. Step responses are:



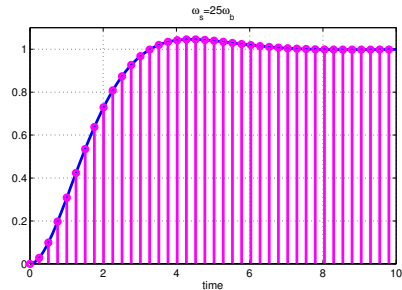
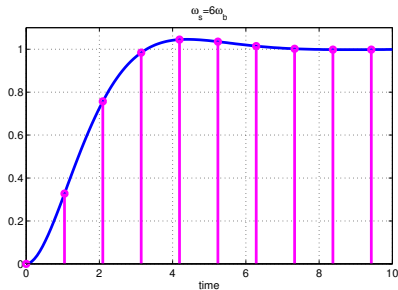
Sampling. Example

First order system $\omega_{s1} = 6\omega_{b1}$ ($T_s \approx T_0 = 1$) or $\omega_{s1} = 25\omega_{b1}$ ($T_s \approx \frac{T_0}{4} = 0.25$)

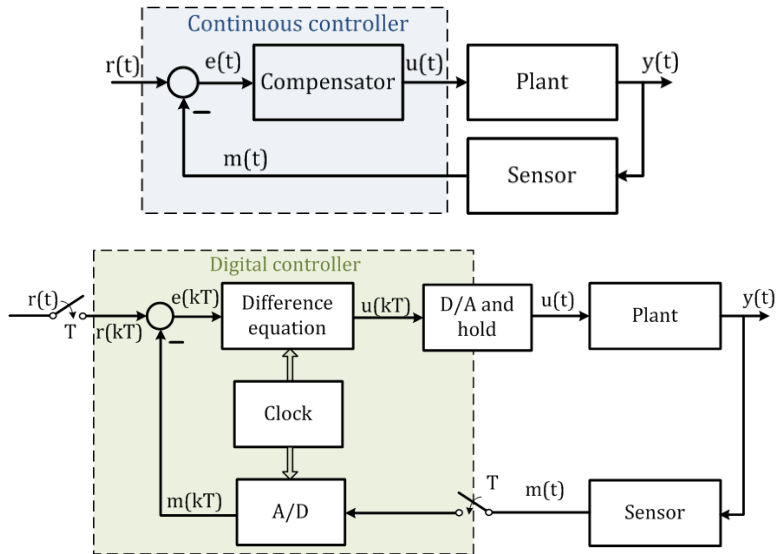


Sampling. Example

Second order system $\omega_{s2} = 6\omega_{b2}$ or $\omega_{s2} = 25\omega_{b2}$

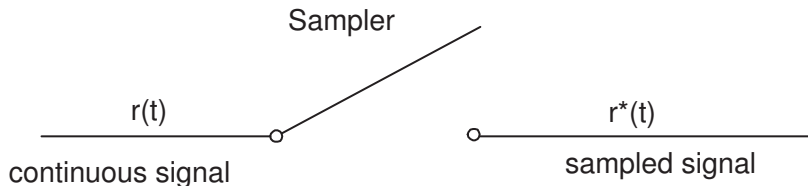


Continuous control vs. digital control - again



Sampled-Data Systems

A **sampler** is basically a switch that closes every T seconds for one instant of time.



The input is $r(t)$ and the output $r^*(t)$ where nT is the current sample time and the current value of $r^*(t)$ is $r(nT)$.

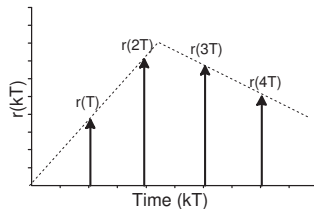
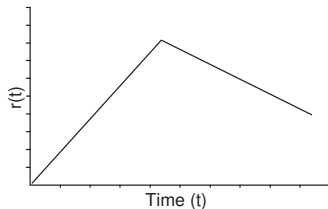
$$r^*(t) = r(nT) \cdot \delta(t - nT)$$

where δ is the impulse function.

Sampled-Data Systems

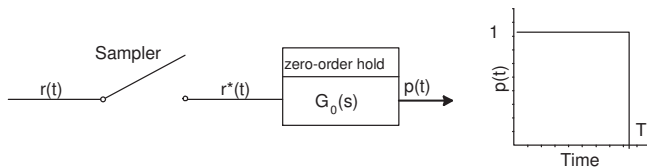
$r^*(t)$ a string of impulses starting at $t = 0$ spaced at T seconds and of amplitude $r^*(kT)$.

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\delta(t - kT)$$



Sampled-Data Systems

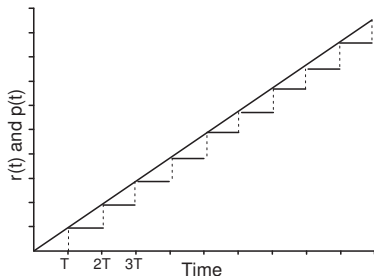
- A digital-to-analog converter: a device that converts the sampled signal $r^*(t)$ to a continuous signal $p(t)$. It can be represented by a **zero-order hold (ZOH)** circuit.
- (Figure) A sampler and zero-order circuit (left); the response of a ZOH to an impulse input $r(kT)$, which equals 1 when $k = 0$ and equals zero when $k \neq 0$ (right):



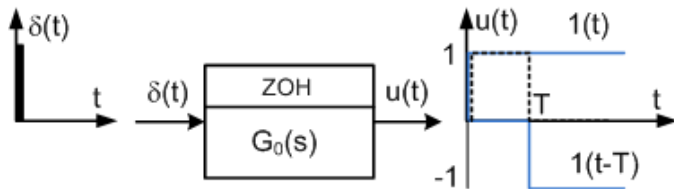
ZOH - takes the value $r(kT)$ and holds it constantly for $kT \leq t \leq (k+1)T$.

Sampled-Data Systems

- A sampler and ZOH can accurately follow the input signal if T is small compared to the transient changes in the signal.
- (Figure) The response of a sampler and zero-order hold for a ramp input $r(t) = t$.



Zero-order hold (ZOH)



(input: $u(t) = \delta(t)$, output: $y(t) = 1(t) - 1(t - T)$, $1(t) =$ unit step)

The transfer function of a ZOH:

$$G_0(s) = \frac{Y(s)}{U(s)} = \frac{1}{s} - \frac{1}{s}e^{-sT} = \frac{1 - e^{-sT}}{s}$$

The z-Transform

The output of an ideal sampler $r^*(t)$ = series of impulses with values $r(kT)$:

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\delta(t - kT), \quad t > 0$$

$$L[r^*(t)] = \sum_{k=0}^{\infty} r(kT)e^{-ksT}, \quad z = e^{sT}$$

⇒ The z-transform:

$$Z[r(t)] = Z[r^*(t)] = \sum_{k=0}^{\infty} r(kT)z^{-k}$$

Z-transform table

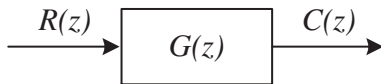
Time domain	Laplace transform	Z-transform
$f(t)$	$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st} dt$	$F(z) = \mathcal{Z}[f(t) _{t=kT}] = \sum_{k=0}^{\infty} f(kT)z^{-k}$
$\delta(t)$	1	1
1	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z - e^{-aT}}{z - 1}$
$\sin at$	$\frac{a}{s^2 + a^2}$	$\frac{z \sin aT}{z^2 - 2z \cos aT + 1}$
$\cos at$	$\frac{s}{s^2 + a^2}$	$\frac{z(z - \cos aT)}{z^2 - 2z \cos aT + 1}$

Properties of z-transform

Property	Discrete time domain	z-domain
	$f(kT)$	$F(z) = \mathcal{Z}[f(kT)]$
Linearity	$af_1(kT) + bf_2(kT)$	$aF_1(z) + bF_2(z)$
Shift right by T	$f((k-1)T)$	$z^{-1}F(z)$
Shift right by nT	$f((k-n)T)$	$z^{-n}F(z)$
Shift left by T	$f((k+1)T)$	$zF(z) - zf(0)$
Shift left by nT	$f((k+n)T)$	$z^n F(z) - \sum_{i=0}^{n-1} f(iT)z^{k-i}$
First difference	$f(kT) - f((k-1)T)$	$(1 - z^{-1})F(z)$
Final value theorem	$f(\infty) = \lim_{k \rightarrow \infty} f(kT)$	$\lim_{z \rightarrow 1} (z-1)F(z)$ if the poles of $(z-1)F(z)$ are inside the unit circle

z transfer function

The transfer function - block diagram



If $C(z)$ is the z-transform of the output signal and the input is $R(z) \Rightarrow$ the transfer function

$$\frac{C(z)}{R(z)} = G(z)$$

Digitizing analog transfer functions - 1 - ZOH

- One approach in digitizing an analog transfer function is transformation from s-domain into z-domain
- Given a tf in s-domain: $G(s)$. The z-transform is obtained as:

$$G(z) = Z \{ G_0(s)G(s) \} = Z \left\{ \frac{1 - e^{-sT}}{s} G(s) \right\}$$

$$G(z) = (1 - z^{-1})Z \left\{ \frac{G(s)}{s} \right\}$$

where $G_0(s)$ - ZOH transfer function and T - sampling time.

- *The following properties of the Laplace and Z transforms are used:*

$$\mathcal{L}[x(t - T)] \rightarrow e^{-sT}X(s); \quad Z[x(t - T)] \rightarrow z^{-1}X(z)$$

Digitizing analog transfer functions - 2 - Euler approximation

- The variable z can be approximated by various substitutions for s in the transfer function of the continuous system.
- Compare: multiplying by s = differentiation; multiplying by z^{-1} = delay of one sampling interval.
- Obs. The time derivative at a moment kT can be approximated by:

$$\left. \frac{de}{dt} \right|_{kT} = \frac{e(kT) - e((k-1)T)}{T}$$

In z -domain, where $e(kT) \rightarrow E(z)$:

$$\frac{E(z) - E(z) \cdot z^{-1}}{T} = E(z) \frac{1 - z^{-1}}{T} \Rightarrow s = \frac{1 - z^{-1}}{T}$$

- Tustin's rule, also called the bilinear transformation, gives a better approximation

$$s = \frac{2(z-1)}{T(z+1)} \quad \text{or} \quad s = \frac{2(1-z^{-1})}{T(1+z^{-1})}$$

- This more complicated substitution allows close approximation of the behavior of the analog controller using a longer sampling interval (T).

Digitizing analog transfer functions. Example

Consider a continuous system having the transfer function:

$$G(s) = \frac{1}{s + 4}$$

- First we shall determine the pulse transfer function using the ZOH method:

$$G_1(z) = Z\{G_0(s)G(s)\} = Z\left\{\frac{1 - e^{-sT}}{s} \frac{1}{s + 4}\right\} = (1 - z^{-1}) \frac{1}{4} Z\left\{\frac{1}{s} - \frac{1}{s + 4}\right\}$$

- Using the z-transform table we determine the z-transforms of basic signals we obtain:

$$G_1(z) = \frac{z - 1}{z} \frac{1}{4} \left(\frac{z}{z - 1} - \frac{z}{z - e^{-4T}} \right) = \frac{1}{4} \frac{1 - e^{-4T}}{z - e^{-4T}}$$

Digitizing analog transfer functions. Example

- Using the Euler substitution $s = \frac{1 - z^{-1}}{T}$, the pulse transfer function is:

$$G_2(z) = \frac{1}{\frac{1 - z^{-1}}{T} + 4} = \frac{T}{1 + 4T - z^{-1}}$$

- Using the Tustin substitution $s = \frac{2(1 - z^{-1})}{T(1 + z^{-1})}$, the pulse transfer function is:

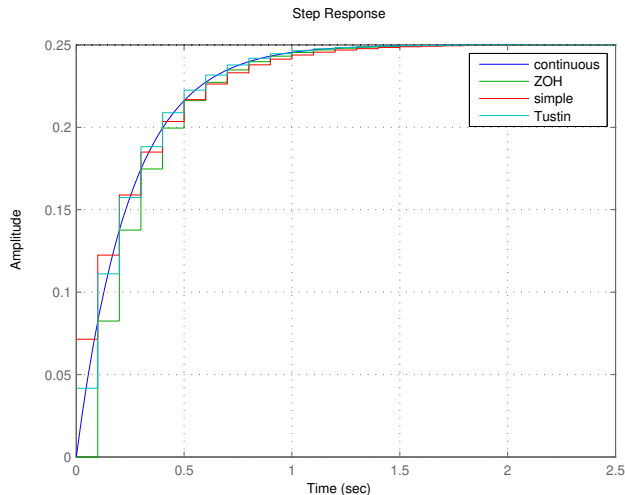
$$G_3(z) = \frac{1}{\frac{2(1 - z^{-1})}{T(1 + z^{-1})} + 4} = \frac{T(1 + z^{-1})}{2 + 4T + (4T - 2)z^{-1}}$$

- For $T = 0.1$, the transfer functions (written in z instead of z^{-1}) in all cases above are:

$$G_1(z) = \frac{0.08242}{z - 0.6703}, \quad G_2(z) = \frac{0.1z}{1.4z - 1}, \quad G_3(z) = \frac{0.04167z + 0.04167}{z - 0.6667}$$

Digitizing analog transfer functions. Example

The simulated unit step response in all three cases is shown in the figure:



Inverse transformation - Infinite power series method

Example If the z-transform of a signal is obtained in the form:

$$X(z) = \frac{z^2}{z^2 - 1.36z + 0.36},$$

by long division we obtain:

$$X(z) = z^2 : (z^2 - 1.36z + 0.36) = 1 + 1.36z^{-1} + 1.5z^{-2} + \dots$$

Compare this result with the definition of the z-transform:

$$X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} = x(0)z^0 + x(T)z^{-1} + x(2T)z^{-2} + \dots$$

Thus:

$$x(0) = 1, \quad x(T) = 1.36, \quad x(2T) = 1.5, \quad \text{etc}$$

Inverse transformation - Difference equation method

- Consider a general discrete system with an input $X(z)$ and an output $Y(z)$, of the form:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + \dots + b_mz^{-m}}{1 + a_1z^{-1} + \dots + a_nz^{-n}}$$

where a_i 's and b_i 's are real coefficients.

- By cross multiplication we obtain:

$$Y(z)(1 + a_1z^{-1} + \dots + a_nz^{-n}) = X(z)(b_0 + b_1z^{-1} + \dots + b_mz^{-m})$$

or

$$Y(z) = -(a_1z^{-1} + a_2z^{-2} + \dots + a_nz^{-n})Y(z) + (b_0 + b_1z^{-1} + b_2z^{-2} + \dots + b_mz^{-m})X(z)$$

$$Y(z) = -a_1z^{-1}Y(z) + a_2z^{-2}Y(z) + \dots + a_nz^{-n}Y(z) + b_0X(z) + b_1z^{-1}X(z) + \dots + b_mz^{-m}X(z)$$

Inverse transformation - Difference equation method

- Using the property:

$$Z[f(kT - T)] = z^{-1}F(z), \text{ where } F(z) = Z[f(kT)]$$

the equation obtained before can be expressed as a difference equation of the form:

$$y(kT) = -a_1y((k-1)T) + a_2y((k-2)T) + \dots + b_0x(kT) + b_1x((k-1)T) + b_2x((k-2)T) + \dots$$

- With a simple notation:

$$y_k = -a_1y_{k-1} + a_2y_{k-2} + \dots + b_0x_k + b_1x_{k-1} + b_2x_{k-2} + \dots$$

where we denoted the value of the signal at the time kT by the subscript k (and in a similar manner, $(k-1)T$ by the subscript $k-1$, etc.)

Obtaining the difference equation. Example

Consider a transfer function:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z}{z - 0.368}$$

We divide the numerator and denominator of the transfer function by z raised to the highest power in the tf and obtain:

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 0.368z^{-1}}$$

or

$$(1 - 0.368z^{-1})Y(z) = X(z) \quad \text{or} \quad Y(z) = 0.368z^{-1}Y(z) + X(z)$$

It can be expressed as a difference equation:

$$y_k = 0.368y_{k-1} + x_k$$

If the first two samples are known, the values for y_k can be determined by an iterative procedure.

Analysis of discrete systems

Mapping between the s-plane and the z-plane

The s-plane and the z-plane are related by a *conformal mapping* specified by the analytic complex function:

$$z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} \cdot e^{j\omega T}$$

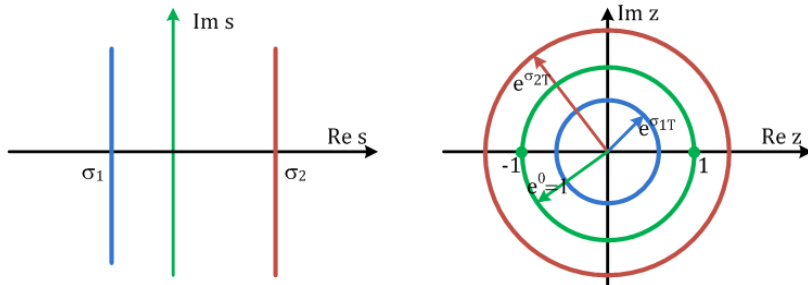
where T is the sampling time and:

$$\operatorname{Re}[s] = \sigma, \quad \operatorname{Im}[s] = j\omega$$

$$|z| = e^{\sigma T}, \quad \angle z = \omega T$$

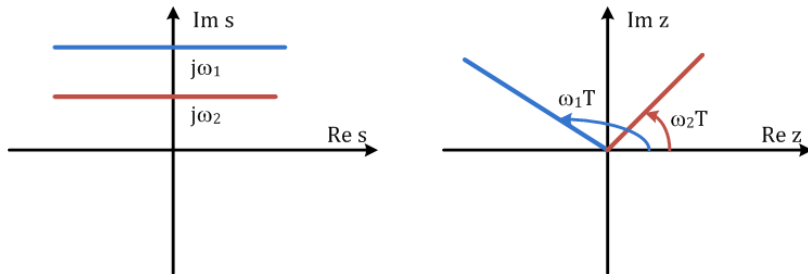
Mapping between the s-plane and the z-plane

- The point $s = 0$ is mapped in the z-plane to $z = e^0 = 1$
- All complex numbers $s = \sigma + j\omega$ with the real part σ constant are mapped into z-plane in circles having the radius $e^{\sigma T}$.



Mapping between the s-plane and the z-plane

- A complex imaginary number $s = +j\omega \rightarrow z = e^{j\omega T}$: a unit length vector at angle ωT . When $\omega T = \pi$, the vector reached the point -1 in z-plane, thus describing half of a circle. When $\omega T = -\pi$, the other half of the circle is obtained.
- Complex numbers $s = \sigma + j\omega$ with the same imaginary part $j\omega$, are mapped to radial lines with constant angle ωT



Stability analysis in z-plane

- A linear continuous system is stable if all the poles of $G(s)$ lie in the left half of the s-plane. The s-plane is related to the z-plane by the transformation:

$$z = e^{sT} = e^{(\sigma + j\omega)T}$$

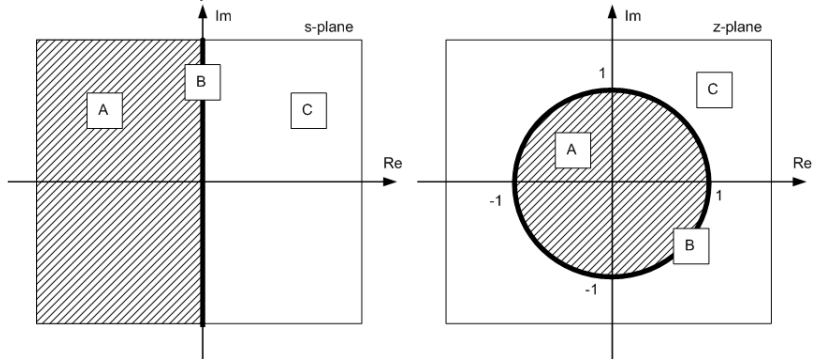
or

$$|z| = e^{\sigma T}, \quad \angle z = \omega T$$

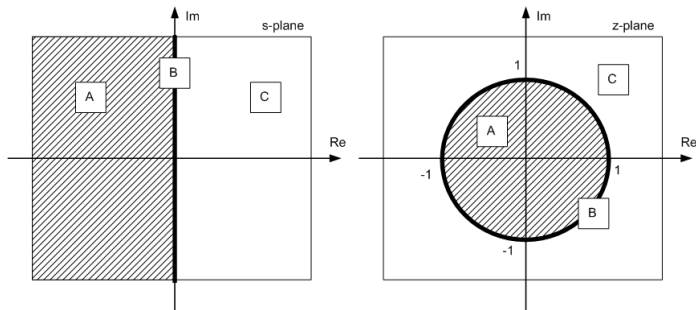
- In the left-hand s-plane, the real part of s , $\sigma < 0 \Rightarrow$ the magnitude of z varies between 0 and 1: $0 < e^{\sigma T} < 1$.
- The imaginary axis of the s-plane corresponds to the unit circle in the z-plane, and the inside of the unit circle corresponds to the left half of the s-plane.

Stability analysis in z-plane

A sampled data system is stable if all the poles of the closed-loop transfer function $G(z)$ lie within the unit circle of the z-plane.



Stability analysis in z-plane



- Points that have negative values of σ (left half s-plane, region A) map into the inside of the unit circle on the z-plane.
- Points on the $j\omega$ axis (region B) have zero values of σ and yield points on the unit circle on the z-plane.
- The points that have positive values of σ are in the right half of the s-plane (region C). The magnitudes of the mapped points are $e^{\sigma T} > 1$, thus they are mapped into points outside the circle on the z-plane.

Discrete systems - stability analysis. Summary

A digital system with a transfer function $G(z)$ is:

- *stable* if all poles of the transfer function are inside the unit circle on the z -plane,
- *unstable* if any pole is outside the unit circle and/or there are poles of multiplicity greater than one on the unit circle,
- *marginally stable* if poles of multiplicity one are on the unit circle and all other poles are inside the unit circle.

Discrete systems - stability analysis. Examples

- A stable system: $G_1(z) = \frac{z}{4z^2 - 1}$ with the poles: $z_1 = \frac{1}{2}$, $z_2 = -\frac{1}{2}$
 $|z_1| = |z_2| = \frac{1}{2} < 1$ - both poles lie inside the unit circle.
- An unstable system: $G_2(z) = \frac{z}{z^2 + 2z + 2}$ with the poles $z_1 = -1 + j$, $z_2 = -1 - j$
 $|z_1| = |z_2| = \sqrt{2} > 1$ - both poles lie outside the unit circle.
- A marginally stable system: $G_3(z) = \frac{z}{2z^2 - z - 1}$ with the poles $z_1 = 1$, $z_2 = -\frac{1}{2}$
 $|z_1| = 1$ and $|z_2| = \frac{1}{2} < 1$ - one pole on the unit circle and the second one inside.
- An unstable system: $G_4(z) = \frac{z}{z^2 + 3z + 2}$ with the poles $z_1 = -1$, $z_2 = -2$
 $|z_1| = 1$ and $|z_2| = 2 > 1$ - one pole on the unit circle and the second one outside.