

Introduction to Control System Modeling

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The Laplace Transform

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

Linearity	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
Constant multiplication	$af(t)$	$aF(s)$
Complex shift theorem	$e^{\pm at} f(t)$	$F(s \pm a)$
Real shift theorem	$f(t-T)$	$e^{-Ts} F(s), T \geq 0$
Scaling theorem	$f\left(\frac{t}{a}\right)$	$aF(as)$
First derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0)$
Second derivative	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$
n-th derivative	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0)$
First integral	$\int_0^t f(t) dt$	$\frac{1}{s} F(s)$

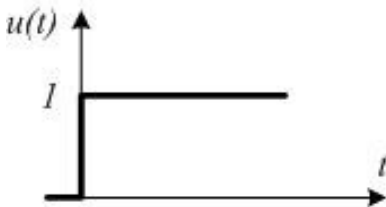
Laplace transform

Table: Laplace transforms of common functions

	$f(t)$	$F(s)$
1	Unit impulse (Dirac) $\delta(t)$	1
2	Unit step $u(t)=1$	$\frac{1}{s}$
3	Unit ramp $v(t)=t$	$\frac{1}{s^2}$
4	e^{-at}	$\frac{1}{s+a}$
5	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
6	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
7	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
8	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

1. The unit step:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

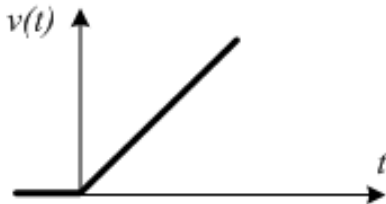


The Laplace transform of the step function:

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

2. The unit ramp

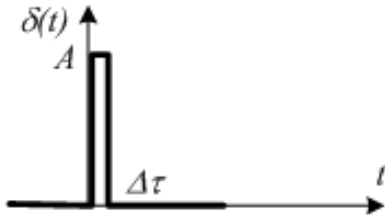
$$v(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$



The Laplace transform of the ramp signal is:

$$\mathcal{L}[v(t)] = \frac{1}{s^2}$$

3. The ideal impulse (Dirac)



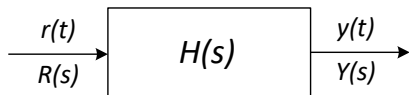
$$\delta(t) = \begin{cases} 0, & t < 0 \text{ and } t > \Delta\tau \\ A, & 0 \leq t \leq \Delta\tau \end{cases}, \quad \lim_{\Delta\tau \rightarrow 0} \int_0^{\Delta\tau} \delta(t) dt = 1$$

The Laplace transform of the unit impulse is:

$$\mathcal{L}[\delta(t)] = 1$$

Transfer function models

Transfer function: The ratio of the Laplace transform of the output signal to the Laplace transform of the input signal, with all the initial conditions assumed to be zero.



$$H(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{r(t)\}}, \text{ în condiții inițiale nule}$$

- For a linear homogeneous differential equation:

$$a_0 r(t) + a_1 \frac{dr(t)}{dt} + \dots + a_m \frac{d^m r(t)}{dt^m} = b_0 y(t) + b_1 \frac{dy(t)}{dt} + \dots + b_n \frac{d^n y(t)}{dt^n}$$

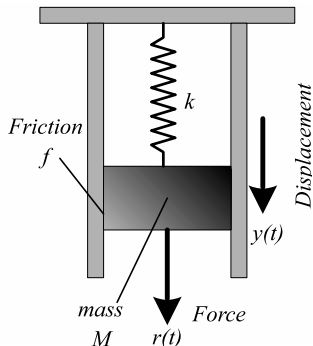
where $r(t)$ and $y(t)$ are the input and output signals

- Apply the Laplace transform (assuming the initial conditions = 0):

$$(a_0 + a_1 s + \dots + a_m s^m) R(s) = (b_0 + b_1 s + \dots + b_n s^n) Y(s)$$

and the transfer function is:
$$H(s) = \frac{Y(s)}{R(s)} = \frac{a_0 + a_1 s + \dots + a_m s^m}{b_0 + b_1 s + \dots + b_n s^n}$$

Example. Spring-mass-damper



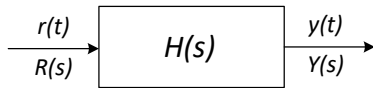
$$M \frac{d^2 y(t)}{dt^2} + f \frac{dy(t)}{dt} + ky(t) = r(t)$$

Apply the Laplace transform for initial conditions = 0 and obtain:

$$Ms^2 Y(s) + fsY(s) + kY(s) = R(s)$$

$$(Ms^2 + fs + k)Y(s) = R(s)$$

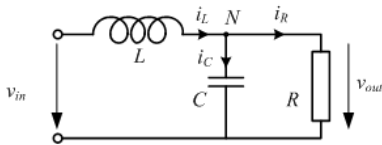
$$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + fs + k}$$



"output = contents x input"

A transfer function $H(s)$ shows how the input is transferred to the output.

Example. Electrical system



Kirchhoff's current law:

$$i_L = i_C + i_R$$

Kirchhoff's voltage law:

$$v_{in} = v_L + v_C$$

$$v_C = v_R = v_{out}$$

Inductor: $\frac{di_L}{dt} = \frac{1}{L}v_L$

Capacitor: $\frac{dv_C}{dt} = \frac{1}{C}i_C$

Resistor: $v_R = Ri_R$

Assume the initial conditions zero, apply the Laplace transform, eliminate all variables except for input and output.

$$\Rightarrow H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{LCs^2 + \frac{L}{R}s + 1} = \frac{R}{RLCs^2 + Ls + R}$$

For detailed calculus see the lecture notes.

The transfer function

For a **real physical** system the function $H(s)$ is a rational polynomial such that

$$H(s) = \frac{N(s)}{D(s)}$$

order of $D(s) \geq$ order of $N(s)$.

The characteristic equation: $D(s) = 0$

Roots of $D(s)$: **poles**.

Roots of $N(s)$: **zeros**

Highest degree of $D(s)$: **system order**

The poles and zeros of $H(s)$ can be complex values, $s = \sigma + j\omega$.

Example 1

Consider a system described by the transfer function

$$H(s) = \frac{s + 2}{s^3 + s^2 + s + 1}$$

- Third order system
- Characteristic polynomial and characteristic equation:

$$D(s) = s^3 + s^2 + s + 1 \quad s^3 + s^2 + s + 1 = 0$$

- The roots of the characteristic polynomial = the system poles:

$$(s + 1)(s^2 + 1) = 0, \text{ and the poles are: } p_1 = -1, p_{2,3} = \pm j$$

- The numerator polynomial has one root = the zero of the system:

$$s + 2 = 0, \text{ and the zero is: } z_1 = -2$$

Example 2

Consider a system described by the transfer function

$$H(s) = \frac{1}{s^2(s^2 + 2s + 2)}$$

- Fourth order system
- Characteristic polynomial and characteristic equation:

$$D(s) = s^2(s^2 + 2s + 2) = s^4 + 2s^3 + 2s^2 \quad s^4 + 2s^3 + 2s^2 = 0$$

- The roots of the characteristic polynomial = the system poles:

$$s^2(s^2 + 2s + 2) = 0, \text{ and the poles are: } p_1 = p_2 = 0, p_{3,4} = -1 \pm j$$

- The numerator is a zero order polynomial $\Rightarrow$ the system has no zeros

System response

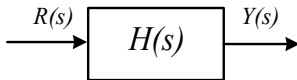


Figure: Block diagram of a system

From the definition of the transfer function:

$$Y(s) = H(s) \cdot R(s) \quad (1)$$

By applying the inverse Laplace transform we obtain:

$$y(t) = \mathcal{L}^{-1}[H(s) \cdot R(s)]. \quad (2)$$

Example. Spring-mass-damper

Spring-mass-damper system with the input $r(t) = \delta(t)$ ideal impulse.

$$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + fs + k}, \quad R(s) = \mathcal{L}[\delta(t)], \quad y(t) = \mathcal{L}^{-1}[H(s) \cdot 1]$$

$$M = 1, f = 3, k = 2$$

$$H(s) = Y(s) = \frac{1}{(s+1)(s+2)}$$

$$y(t) = e^{-t} - e^{-2t}$$

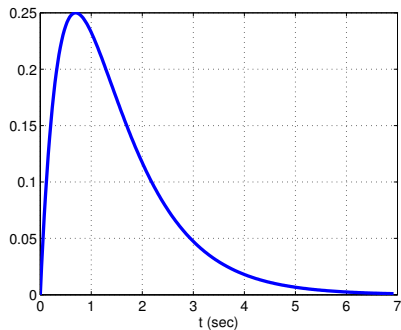
$$M = 1, f = 1, k = 2$$

$$H(s) = Y(s) = \frac{K}{s^2 + s + 2}$$

$$y(t) = \frac{2}{\sqrt{7}} e^{-t/2} \sin\left(\frac{\sqrt{7}}{2} t\right)$$

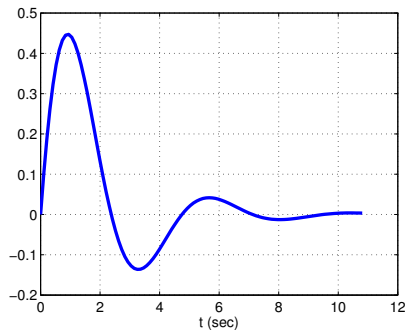
Example. Spring-mass-damper

$$y(t) = e^{-t} - e^{-2t}$$



Overdamped response.

$$y(t) = \frac{2}{\sqrt{7}} e^{-t/2} \sin\left(\frac{\sqrt{7}}{2} t\right)$$

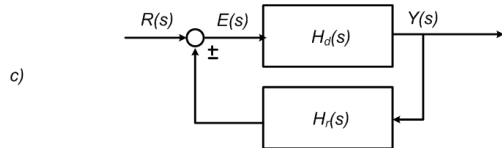
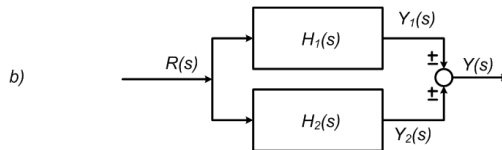


Underdamped response

Block diagram models

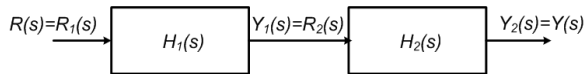
Block Diagram Models

- Basic connections: a) series, b) parallel and c) feedback.



Block Diagram Models

■ Series (cascade) connection



$$Y(s) = Y_2(s) = H_2(s)R_2(s) = H_2(s)Y_1(s) = H_2(s)H_1(s)R_1(s)$$

$$Y(s) = H_2(s)H_1(s)R(s)$$

■ The equivalent transfer function from the input $R(s)$ to the output $Y(s)$:

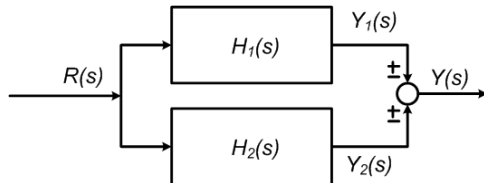
$$\text{series: } H(s) = \frac{Y(s)}{R(s)} = H_1(s)H_2(s)$$

■ for n system in series:

$$H(s) = \prod_{j=1}^n H_j(s)$$

Block Diagram Models

■ Parallel connection



$$H_1(s) = \frac{Y_1(s)}{R(s)}; \quad H_2(s) = \frac{Y_2(s)}{R(s)}; \quad Y(s) = \pm Y_1(s) \pm Y_2(s) = \pm H_1(s)R(s) \pm H_2(s)R(s)$$

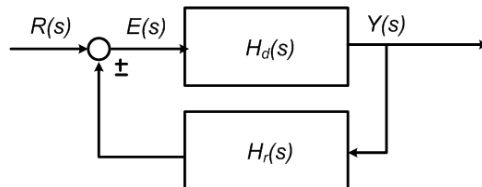
paralell:
$$H(s) = \frac{Y(s)}{R(s)} = \pm H_1(s) \pm H_2(s)$$

■ For n systems in parallel:

$$H(s) = \sum_{j=1}^n (\pm H_j(s))$$

Block Diagram Models

■ Feedback connection



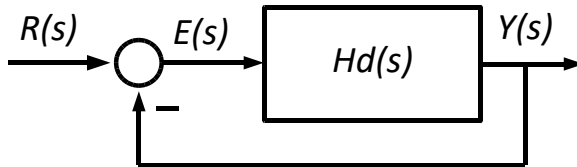
$$E(s) = R(s) \pm H_r(s) \cdot Y(s)$$

$$Y(s) = H_d(s) \cdot E(s) = H_d(s) \cdot (R(s) \pm H_r(s) \cdot Y(s))$$

feedback:
$$H(s) = \frac{Y(s)}{R(s)} = \frac{H_d(s)}{1 \mp H_d(s) \cdot H_r(s)}$$

Block Diagram Models

Unity negative feedback

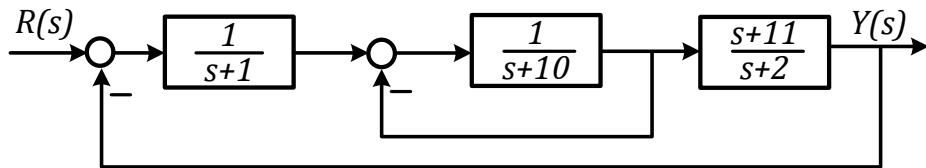


$$H_r(s) = 1$$

$$H(s) = \frac{Y(s)}{R(s)} = \frac{H_d(s)}{1 + H_d(s)}$$

Example

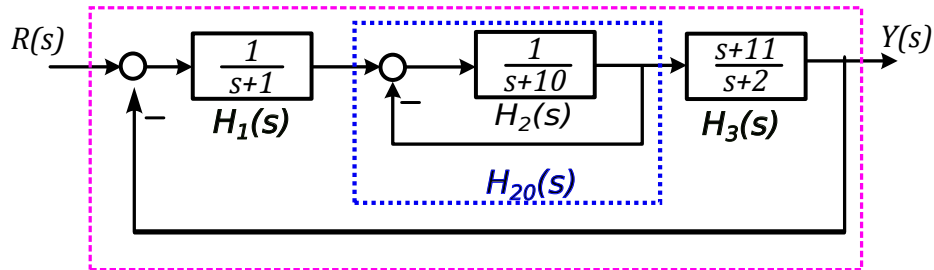
Determine the overall transfer function, from the input $R(s)$ to the output $Y(s)$ for the following block diagram:



Denote:

$$H_1(s) = \frac{1}{s+1}, \quad H_2(s) = \frac{1}{s+10}, \quad H_3(s) = \frac{1}{s+1}$$

Example

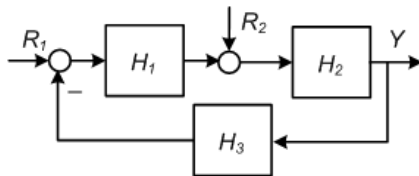


$$H_{20}(s) = \frac{H_2(s)}{1 + H_2(s)} = \frac{\frac{1}{s+10}}{1 + \frac{1}{s+10}} = \frac{1}{s+11}$$

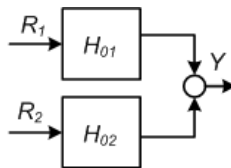
$$H_0 = \frac{H_1(s)H_{20}(s)H_3(s)}{1 + H_1(s)H_{20}(s)H_3(s)} = \frac{\frac{1}{s+1} \frac{1}{s+11} \frac{s+11}{s+2}}{1 + \frac{1}{s+1} \frac{1}{s+11} \frac{s+11}{s+2}}$$

$$H_0(s) = \frac{\frac{1}{(s+1)(s+2)}}{1 + \frac{1}{(s+1)(s+2)}} = \frac{1}{(s+1)(s+2) + 1} = \frac{1}{s^2 + 3s + 3}$$

The overlap of signals



a)



b)

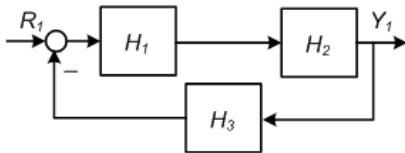
$$Y(s) = R_1(s) \cdot H_{01}(s)|_{R_2(s)=0} + R_2(s) \cdot H_{02}(s)|_{R_1(s)=0}$$

$$Y(s) = \underbrace{\frac{H_1 H_2}{1 + H_1 H_2 H_3}}_{H_{01}} \cdot R_1(s) + \underbrace{\frac{H_2}{1 + H_1 H_2 H_3}}_{H_{02}} \cdot R_2(s)$$

The calculus for H_{01} and H_{02} is given below:

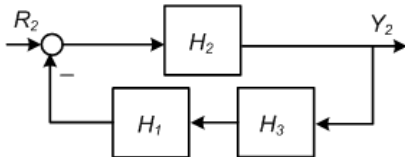
The overlap of signals

$$R_2(s) = 0$$



$$H_{01}(s) = \frac{H_1 H_2}{1 + H_1 H_2 H_3}$$

$$R_1(s) = 0$$



$$H_{02}(s) = \frac{H_2}{1 + H_1 H_2 H_3}$$