

Analysis of Linear Continuous Systems

Paula Raica

Department of Automation
Dorobantilor 71-73, room C21,
email: Paula.Raica@aut.utcluj.ro

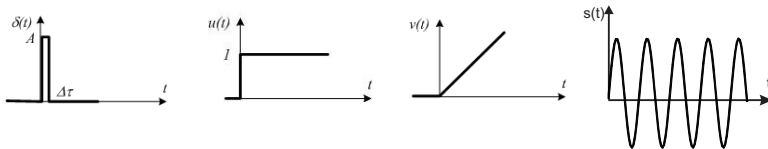
Alexandru Codrean

Department of Automation
Dorobantilor 71-73, room C23,
email: alexandru.codrean@aut.utcluj.ro

Technical University of Cluj-Napoca

System Analysis

- First step: derive a mathematical model
- Various methods are available for analysis
- Performance is analyzed based on various input test signals
- The aim of analysis: **study the system behavior in transient and steady-state when the system model and input are known.**
- Typical test signals: impulse, step, ramp, sinusoidal.



- Transient regime (state), steady-state, periodic regime (state)

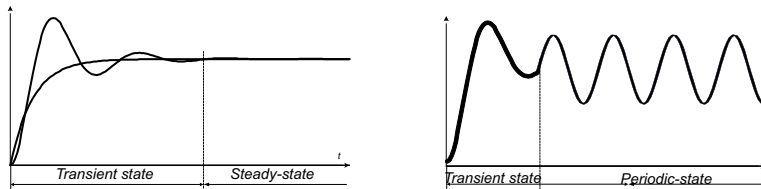
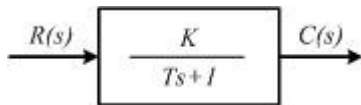


Figure: System response

- The system is decomposed in simple elements of at most second order and the effects of each element are analyzed
- The behavior of simple systems can be studied using some characteristic parameters:
 - Time constants, T
 - Time delay constant, T_m
 - Damping factor ζ
 - Natural frequency ω_n
 - Gain constant, K

First-order systems



The transfer function is given by:

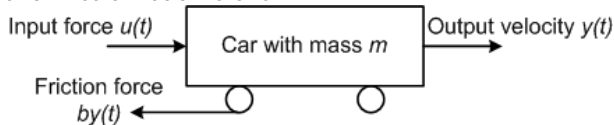
$$H(s) = \frac{C(s)}{R(s)} = \frac{K}{Ts + 1}$$

- K - the gain constant
- T - the time constant, $T > 0$

We will analyze the system responses to inputs as the unit step, unit ramp and unit impulse functions. The initial conditions are assumed to be zero.

First-order system. Example. Mechanical system

- A car with mass m moving on a line
- $u(t)$ is an external force = the input signal
- $y(t)$ is the velocity of the car = the output signal
- Friction force: b = the friction coefficient.



- The differential equation relating the input and output:

$$m \frac{dy(t)}{dt} + by(t) = u(t) \quad | \mathcal{L}, (IC = 0)$$

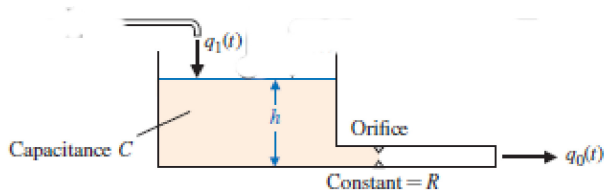
$$\Rightarrow msY(s) + bY(s) = U(s) \Rightarrow Y(s)(ms + b) = U(s)$$

- Transfer function: $H(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms + b} = \frac{\frac{1}{b}}{\frac{m}{b}s + 1}$

- Gain $K = 1/b$, time constant $T = m/b$.

First-order systems. Examples

Hydraulic system example:



$$H(s) = \frac{h(s)}{q_1(s)} = \frac{R}{RCs + 1}.$$

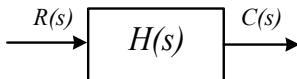
Other numerical examples:

$$H_1(s) = \frac{2}{s + 2} = \frac{1}{\frac{1}{2}s + 1}, \quad \Rightarrow \quad K = 1, \quad T = \frac{1}{2}$$

$$H_2(s) = \frac{4}{3s + 2} = \frac{2}{\frac{3}{2}s + 1}, \quad \Rightarrow \quad K = 2, \quad T = \frac{3}{2}$$

First-order systems. System response

We know: the input $r(t)$ and the transfer function $H(s) = \frac{K}{Ts+1}$.



We want to determine the time evolution of the output (system response), using the Laplace transform:

$$R(s) = \mathcal{L}[r(t)]$$

$$C(s) = H(s) \cdot R(s) \quad \Rightarrow \quad c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}[H(s) \cdot R(s)]$$

First-order systems. Unit-step response

$$r(t) = 1 \Rightarrow R(s) = \frac{1}{s}, \quad C(s) = \frac{K}{Ts + 1} \frac{1}{s}$$

$$c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{K}{s} - \frac{KT}{Ts + 1}\right] = K(1 - e^{-t/T}), \quad (t \geq 0)$$

- For $t \rightarrow \infty \Rightarrow e^{-t/T} \rightarrow 0$ and $c(\infty) = K$ (steady-state)
- At $t = T$ the value of $c(t)$ is $0.632K$, or the response has reached 63.2% of its total change:

$$c(T) = K(1 - e^{-1}) = 0.632K$$

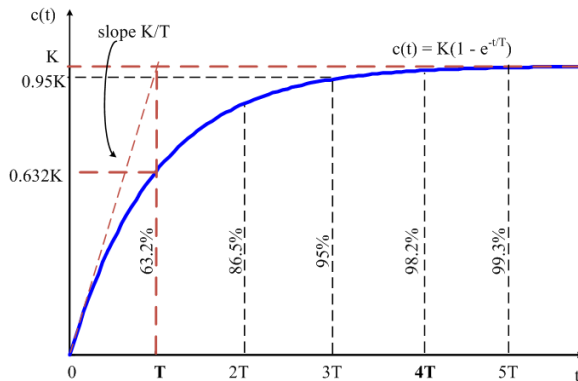
- The slope of the tangent at $t = 0$ is K/T :

$$\left.\frac{dc(t)}{dt}\right|_{t=0} = \frac{K}{T} e^{-t/T} \Big|_{t=0} = \frac{K}{T}$$

- At $t = 4T$ the response has reached 98% of its final value:

$$c(4T) = K(1 - e^{-4T/T}) = 0.982K$$

First-order systems. Unit-step response



For $t \geq 4T$ the response remains within 2% of the final value.

The settling time is:

$$t_s = 4T$$

First-order systems. Step response

The smaller the time constant - the faster the response.

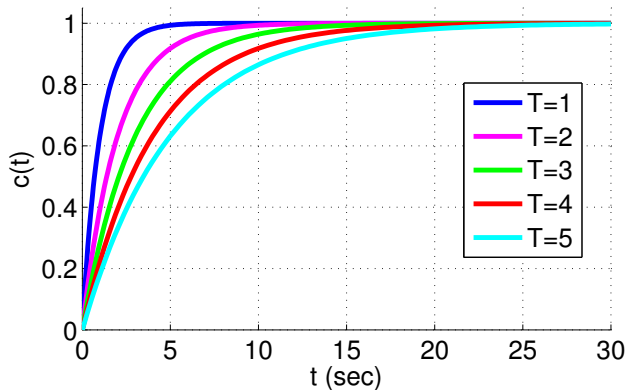


Figure: First-order system step response for various time constants

First-order systems. Unit-ramp response

$$r(t) = t, \Rightarrow R(s) = \frac{1}{s^2}, \quad C(s) = \frac{K}{Ts + 1} \frac{1}{s^2}$$

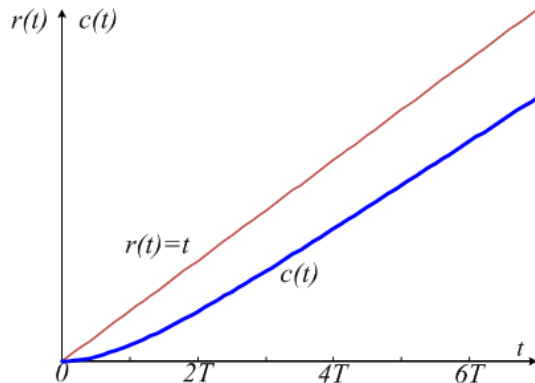
Expanding $C(s)$ into partial fraction gives

$$\begin{aligned} c(t) &= \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1} \left[K \left(\frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1} \right) \right] \\ &= K(t - T + Te^{-t/T}), \quad (t \geq 0) \end{aligned}$$

When time approaches infinity $t \rightarrow \infty$, the system response will follow asymptotically a line with the equation:

$$c(t) = K(t - T)$$

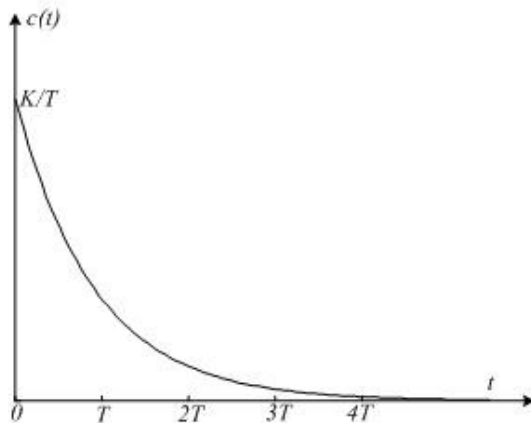
Unit-ramp response



$$e^{-4T/T} = e^{-4} = 0.0183, \Rightarrow t_s = 4T$$

First-order systems. Unit-impulse response

$$r(t) = \delta(t), \Rightarrow R(s) = 1, \quad C(s) = \frac{K}{Ts + 1}, \Rightarrow c(t) = \frac{K}{T}e^{-t/T}, \quad (t \geq 0)$$

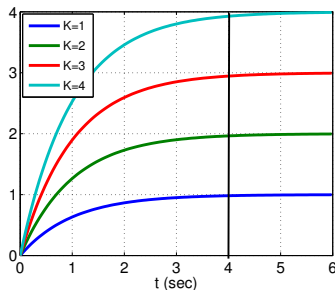


First-order systems. System gain - example

Consider a first-order system with the transfer function:

$$H(s) = \frac{K}{s + 1}$$

- $T = 1$ and $t_s = 4T = 4\text{sec}$, for any value of K .
- The steady-state value of the output, for a unit step input is K . Step response for various values of K :



First-order systems. Influence of the system gain

Consider:

- a system (any linear system) with a transfer function $H(s)$ and the gain equal to 1 *and*
- a system with the transfer function $H_k(s) = kH(s)$

The step responses are:

- for $H(s)$:

$$c(t) = L^{-1}[H(s)R(s)] = L^{-1}\left[\frac{H(s)}{s}\right]$$

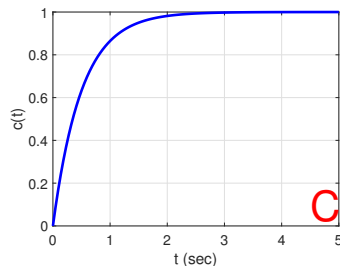
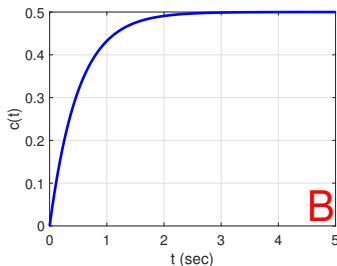
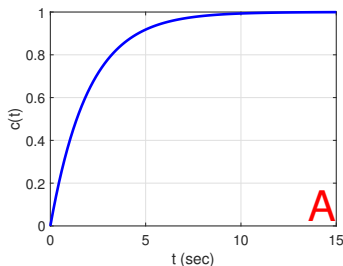
- for $H_k(s)$:

$$c_k(t) = L^{-1}[H_k(s) \cdot R(s)] = L^{-1}\left[\frac{k \cdot H(s)}{s}\right] = k \cdot L^{-1}\left[\frac{H(s)}{s}\right] = k \cdot c(t)$$

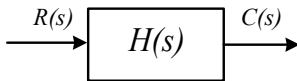
Quiz

Match the unit step responses in the figure with the transfer functions:

$$G_1(s) = \frac{1}{s+2}, \quad , G_2(s) = \frac{2}{s+2}, \quad , G_3(s) = \frac{1}{2s+1}$$



Second-order systems



$$H(s) = \frac{C(s)}{R(s)} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} = \frac{K \cdot \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n - the natural frequency, ζ - the damping factor, K the gain.

$$\omega_n > 0, \quad \zeta \geq 0$$

Second-order systems. Examples

The general form:

$$H(s) = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} = \frac{K \cdot \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Examples:

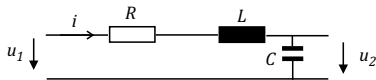
$$\blacksquare H_1(s) = \frac{1}{s^2 + s + 1}, \quad \Rightarrow \quad \frac{1}{\omega_n^2} = 1, \quad \frac{2\zeta}{\omega_n} = 1$$

$$\Rightarrow \omega_n = 1; \quad \zeta = \frac{1}{2}, \quad K = 1$$

$$\blacksquare H_2(s) = \frac{4}{s^2 + 2s + 2} = \frac{2}{\frac{1}{2}s^2 + s + 1}$$

$$\Rightarrow \frac{1}{\omega_n^2} = \frac{1}{2}; \Rightarrow \omega_n = \sqrt{2}, \quad \frac{2\zeta}{\omega_n} = \frac{2\zeta}{\sqrt{2}} = 1 \Rightarrow \zeta = \frac{\sqrt{2}}{2}, \quad K = 2$$

Second-order systems. Electrical circuit example



Kirchhoff voltage law for the loop of the circuit:

$$u_1(t) = i(t) \cdot R + u_L(t) + u_2(t).$$

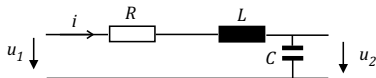
Current-voltage relations for an inductor and a capacitor

$$u_L(t) = L \cdot \frac{di(t)}{dt}, \quad i(t) = C \cdot \frac{du_2(t)}{dt}.$$

By replacing u_L and i we obtain

$$u_1(t) = R \cdot C \cdot \frac{du_2(t)}{dt} + L \cdot C \cdot \frac{d^2 u_2(t)}{dt^2} + u_2(t).$$

Second-order systems. Electrical circuit example



By applying the Laplace transform with zero initial we get transfer function

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{1}{LCs^2 + RCs + 1}.$$

The parameters of the system are

$$\frac{1}{\omega_n^2} = LC \Rightarrow \omega_n = \frac{1}{\sqrt{LC}}, \quad \frac{2\zeta}{\omega_n} = \frac{2\zeta}{\frac{1}{\sqrt{LC}}} = RC \Rightarrow \zeta = \frac{RC}{2\sqrt{LC}}, \quad K = 1$$

Second-order systems

- For the analysis of the second-order system consider $K = 1$.
- A different gain will not change the characteristics of the transient response, it will influence only the magnitude of the response (proportional to K) and the steady-state value.
- The poles of $H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, are the solutions of the characteristic equation:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Delta = 4\zeta^2\omega_n^2 - 4\omega_n^2 = 4\omega_n^2(\zeta^2 - 1)$$

The poles are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

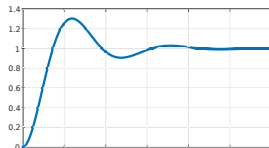
Second-order systems - 3 types of response

underdamped system

$0 < \zeta < 1 \Rightarrow \Delta < 0 \Rightarrow$ complex conjugate poles

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2}j,$$

response: damped oscillations

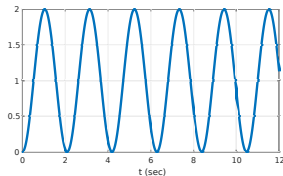


undamped system

$\zeta = 0 \Rightarrow$ poles on the imaginary axis:

$$s_{1,2} = \pm\omega_n j,$$

response: undamped (sustained) oscillations

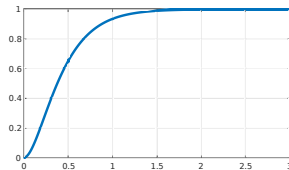


overdamped system

$\zeta \geq 1 \Rightarrow \Delta \geq 0 \Rightarrow$ real poles

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1},$$

response: no oscillations



Step-response of underdamped systems

$$r(t) = 1 \Rightarrow R(s) = \frac{1}{s}, \quad C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Underdamped system: $0 < \zeta < 1$. Poles are complex $s_{1,2} = -\zeta\omega_n \pm \omega_n j\sqrt{1 - \zeta^2}$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ - **damped natural frequency**.

$$\mathcal{L}^{-1}[C(s)] = c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \cdot \sin\left(\omega_d t + \arctan\frac{\sqrt{1 - \zeta^2}}{\zeta}\right)$$

Step-response of underdamped systems

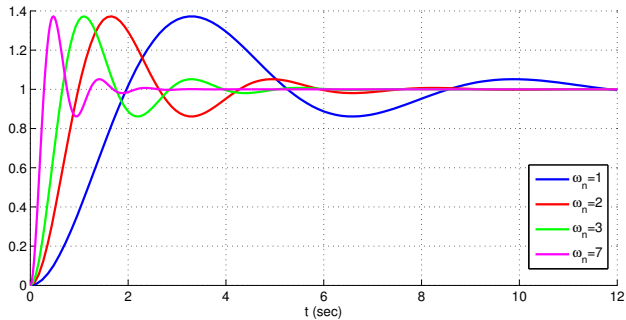


Figure: Step response of an underdamped second order system for various values of ω_n and constant ζ

Step-response of underdamped systems

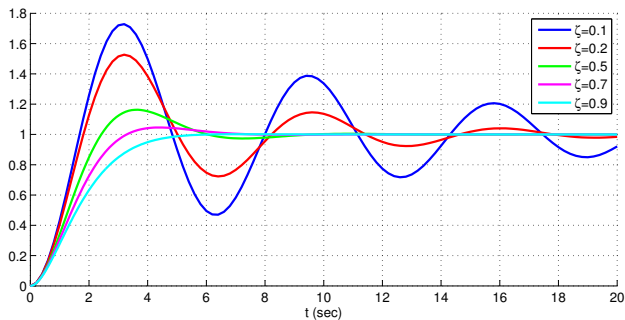


Figure: Step response of an underdamped second order system for various values of ζ and constant ω_n

Step-response of undamped systems

Undamped system: $\zeta = 0$. Imaginary poles $s_{1,2} = \pm j\omega_n$

$$H(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}, \quad R(s) = \frac{1}{s}, \quad C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

Step response:

$$c(t) = 1 - \cos \omega_n t, \quad (t \geq 0)$$

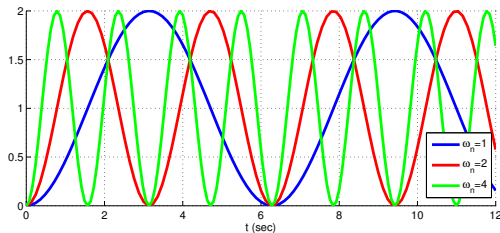


Figure: Step response of an undamped second order system for various values of ω_n

Step-response of critically damped systems

Critically damped system: $\zeta = 1$. Poles are real and equal: $s_{1,2} = -\omega_n$

$$H(s) = \frac{\omega_n^2}{(s + \omega_n)^2}, \quad R(s) = \frac{1}{s}, \quad C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

Step response:

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \quad (t \geq 0)$$

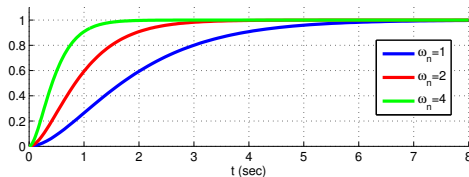


Figure: Step response of critically damped second order system for various values of ω_n

Step-response of overdamped systems

Overdamped system: $\zeta > 1$.

Poles are real and **negative**: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.

$$C(s) = \frac{\omega_n^2}{s(s - s_1)(s - s_2)}$$

The step response: $c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{s_1 t}}{s_1} - \frac{e^{s_2 t}}{s_2} \right)$

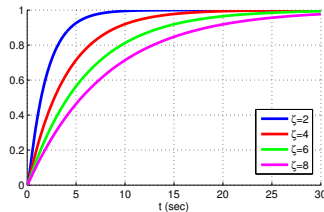


Figure: Step response of an overdamped 2nd order system for various ζ

Step-response of second-order systems

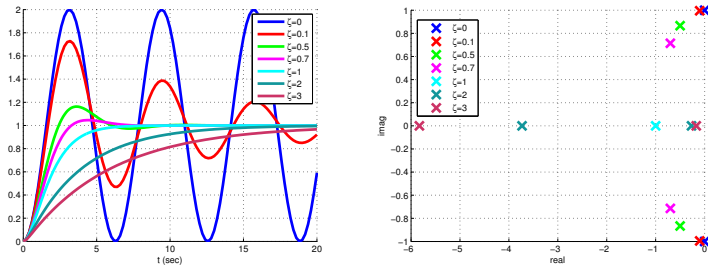


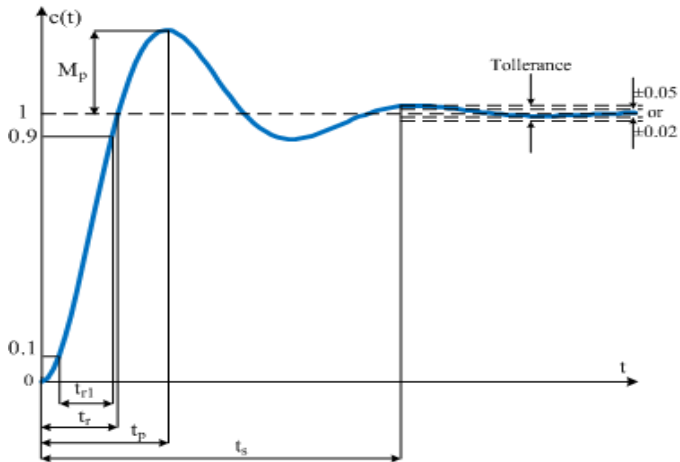
Figure: Step response of second-order systems for various values of ζ and the system poles

See also the RLC circuit example (solved exercise 4.1) from Systems theory Exercise handbook.

Transient-response specifications

Rise

time, Peak time, Maximum overshoot, Settling time



Transient-response specifications

1. **Rise time, t_r** : the time required for the response to rise from 10% to 90%, or 0% to 100% of its final value.
2. **Peak time, t_p** : the time required for the response to reach the first, or maximum, peak. (Not defined for overdamped systems.)
3. **Maximum overshoot M_p** : the maximum peak value of the response curve measured from the steady-state value of the response. The percent overshoot is expressed as percentage of the steady-state value ($M_{p\%}$):

$$M_{p\%} = \frac{c(t_p) - c(\infty)}{c(\infty)} \cdot 100\%$$

where $c(\infty)$ is the final (steady-state) value of the output.

4. **Settling time, t_s** : the time required for the response curve to reach and stay within a range about the final value, usually 2% or 5% of the final value.

Rise time

Obtain the rise time t_r by letting $c(t_r) = 1$ or

$$c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \cdot \sin\left(\omega_d t_r + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 1$$

or

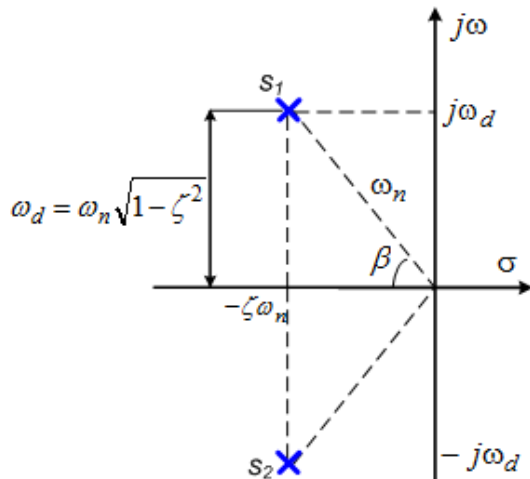
$$\sin\left(\omega_d t_r + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0$$

$$t_r = \frac{1}{\omega_d} \cdot \left(\pi - \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \frac{\pi - \beta}{\omega_d}$$

β = the angle between the negative real axis and the line that connects the pole s_1 to the origin (see the next figure).

Poles of 2nd order system

Important picture !!!



Peak time

Obtain the peak time by differentiating $c(t)$ with respect to time and letting the derivative equal zero:

$$\left. \frac{dc(t)}{dt} \right|_{t=t_p} = \sin(\omega_d t_p) \cdot \frac{\omega_n}{\sqrt{1 - \zeta^2}} \cdot e^{-\zeta \omega_n t_p} = 0$$

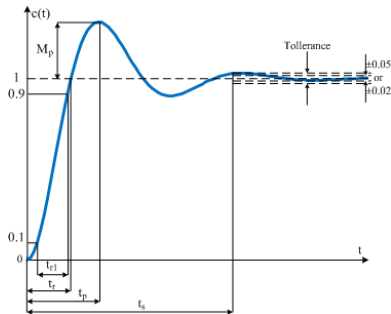
$$\sin(\omega_d t_p) = 0$$

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \dots,$$

$$t_p = \frac{\pi}{\omega_d}$$

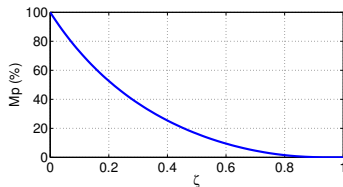
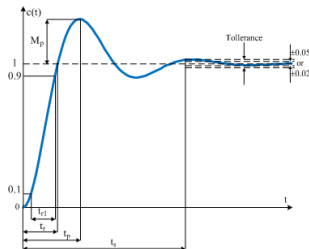
where:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$



Percent overshoot

M_p occurs at the peak time or at $t = t_p = \frac{\pi}{\omega_d}$.



$$M_p = c(t_p) - c(\infty) = c(t_p) - 1 = -\frac{e^{-\zeta\omega_n\pi/\omega_d}}{\sqrt{1-\zeta^2}} \sin(\omega_d\pi/\omega_d + \beta)$$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

Percent overshoot:

$$M_{p\%} = \frac{c(t_p) - c(\infty)}{c(\infty)} \cdot 100\% = \frac{c(t_p) - 1}{1} \cdot 100\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \cdot 100\%$$

Settling time

$$c(t) = 1 - e^{-\zeta\omega_n t} / \sqrt{1 - \zeta^2} \cdot \sin(\omega_d t + \beta)$$

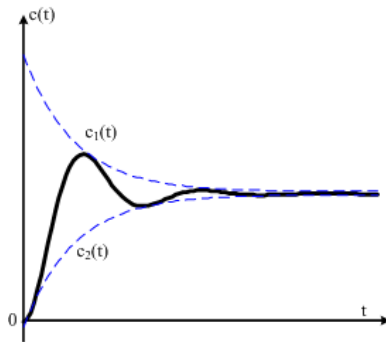
The envelope curves:

$$c_{1,2}(t) = 1 \pm e^{-\zeta\omega_n t} / \sqrt{1 - \zeta^2}$$

$c_1(t)$, $c_2(t)$ and $c(t)$ will reach 2% from the final value approximately when

$$e^{-\zeta\omega_n t_s} < 0.02, \text{ or } \zeta\omega_n t_s \cong 4$$

$$t_s = \frac{4}{\zeta\omega_n}$$



Example

Consider

the closed-loop system with the transfer function:

$$H(s) = \frac{25}{s^2 + 6s + 25} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Calculate:

1 The system poles are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2}j = -3 \pm 4j$$

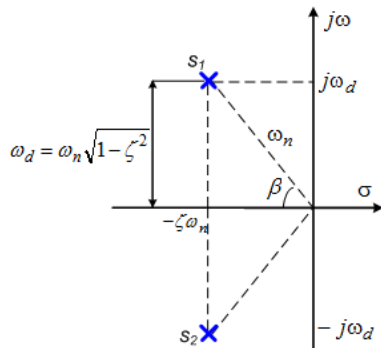
2 The damped natural frequency (the imaginary part of the poles) is:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 5\sqrt{1 - 0.6^2} = 4$$

and the poles negative real part:

$$-\zeta\omega_n = -3.$$

Example



$$\beta = \arctan \frac{\omega_d}{\zeta \omega_n} = 0.93$$

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - 0.93}{4} = 0.55 \text{sec}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.78 \text{sec}$$

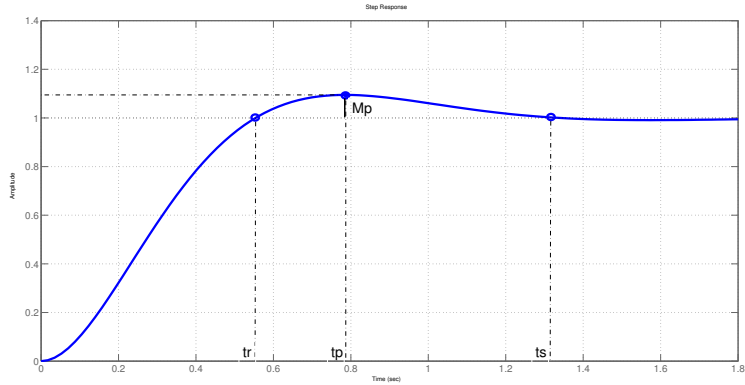
$$M_p = e^{-\pi \zeta / \sqrt{1 - \zeta^2}} = 0.095$$

$$M_p(\%) = 9.5\%$$

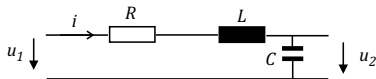
$$t_s = \frac{4}{\zeta \omega_n} = \frac{4}{3} = 1.33 \text{sec}$$

Example

The step response of the system. The values of the system parameters can be seen from the plot.



RLC electrical circuit exercise



Consider the transfer function for the previous RLC circuit example:

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{1}{LCs^2 + RCs + 1}.$$

with the parameters $R = 50\Omega$, $L = 5mH$, $C = 2\mu F$.

- Calculate the rise time, peak time, overshoot and settling time analytically on paper.
- Simulate the system's step response in Matlab and check you calculations.

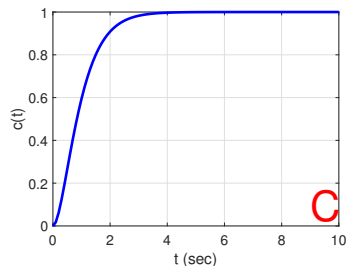
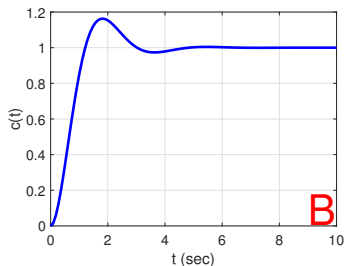
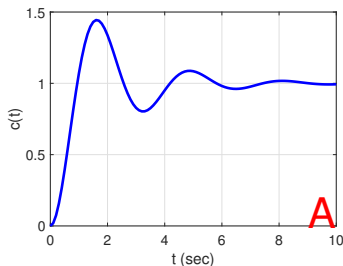
Quiz

Match the unit step responses with the transfer functions:

$$G_1(s) = \frac{4}{s^2 + 4s + 4},$$

$$, G_2(s) = \frac{4}{s^2 + 2s + 4},$$

$$, G_3(s) = \frac{4}{s^2 + s + 4}$$



Quiz

Match the unit step responses with the transfer functions:

$$G_4(s) = \frac{4}{s^2 + 4}, \quad G_5(s) = \frac{1}{s^2 + 1}, \quad G_6(s) = \frac{4}{s^2 + 5s + 4}$$

