

Analysis of Linear Continuous Systems

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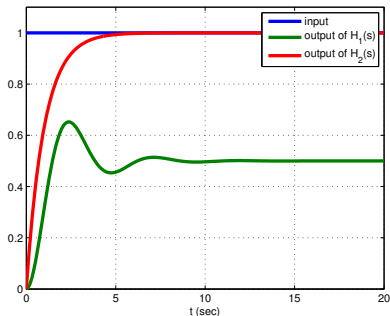
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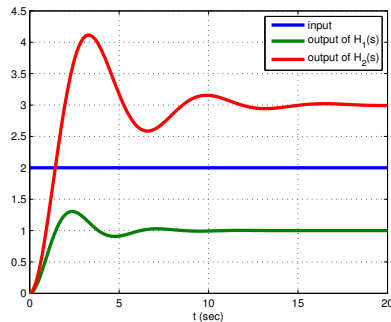
Universitatea Tehnică din Cluj-Napoca

The static gain (DC gain) of a stable system

The **static gain** or the **DC gain** of the system is the ratio of the steady-state output of the system to its steady-state input.

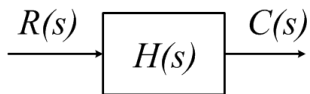


- $H_1(s)$: $DCgain = 0.5/1 = 0.5$
- $H_2(s)$: $DCgain = 1/1 = 1$



- $H_1(s)$: $DCgain = 1/2 = 0.5$
- $H_2(s)$: $DCgain = 3/2 = 1.5$

The static gain (DC gain) of a stable system



- For any constant (step) input $r(t) = A$, the output is:

$$C(s) = H(s) \frac{A}{s}$$

- The steady-state value of the output is:

$$c(\infty) = \lim_{t \rightarrow \infty} c(t) = \lim_{s \rightarrow 0} sC(s) = \lim_{s \rightarrow 0} sH(s) \frac{A}{s} = A \lim_{s \rightarrow 0} H(s)$$

- The DC gain is:

$$DCgain = \frac{c(\infty)}{A} = \lim_{s \rightarrow 0} H(s)$$

DC gain. Example

Consider a system having the transfer function:

$$H(s) = \frac{s + 10}{(s + 2)(s + 4)(s^2 + s + 5)}$$

- The DC gain is:

$$DCgain = \lim_{s \rightarrow 0} \frac{s + 10}{(s + 2)(s + 4)(s^2 + s + 5)} = \frac{10}{2 \cdot 4 \cdot 5} = \frac{1}{4} = 0.25$$

- If the input is a unit step $r(t) = 1$, the steady-state value of the output will be:

$$c(\infty) = DCgain = 0.25$$

- If the input is a constant signal $r(t) = 5$, the steady-state value of the output is:

$$c(\infty) = 0.25 \cdot 5 = 1.25$$

Additional zero

Consider a system with the transfer function $H(s)$.

- The system step response: $c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right]$
- Add a zero at $-a$ and divide the transfer function with a (the DC gain of the new system is unchanged):

$$H_z(s) = \frac{s+a}{a}H(s) = \frac{s}{a}H(s) + H(s)$$

- The step response of the system $H_z(s)$ is:

$$c_z(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{s}{a}H(s) + H(s)\right)\right]$$

$$c_z(t) = \mathcal{L}^{-1}\left[\frac{1}{a}sC(s) + C(s)\right] = \frac{1}{a}\dot{c}(t) + c(t)$$

- If a is small $\Rightarrow 1/a$ is large $\Rightarrow$ the step response of $H_z(s)$ will increase with the quantity $1/a \cdot \dot{c}(t)$.
- Addition of a zero $\Rightarrow$ the increase of the overshoot.

Additional zero. Example

Consider the system with the transfer function:

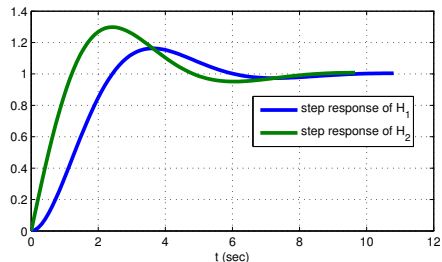
$$\text{System 1: } H_1(s) = \frac{1}{s^2 + s + 1}$$

We add a zero at -1 and obtain:

$$\text{System 2: } H_2(s) = \frac{s + 1}{s^2 + s + 1}$$

Compare:

- System 1: no zeros (blue line)
- System 2: a zero at -1 (green line)



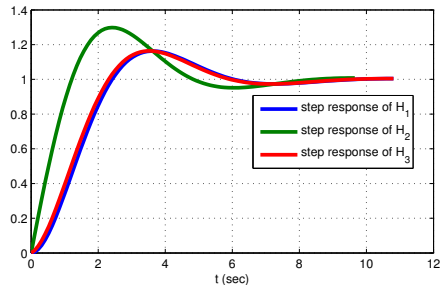
Additional zero. Example

Add a zero at -10 (and divide the transfer function by 10 to keep the same DC gain)

$$\text{System 3: } H_3(s) = \frac{0.1(s + 10)}{s^2 + s + 1}$$

Compare:

- System 1: no zeros (blue line)
- System 2: a zero at -1 (green line)
- System 3: a zero at -10 (red line)



Higher-order systems

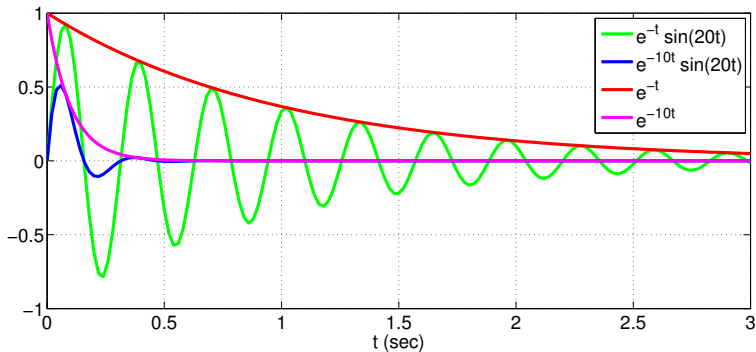
Consider a system $H(s)$, with unit step input $R(s) = 1/s$ and output $C(s)$.

$$C(s) = H(s)R(s) = \frac{H(s)}{s} = \frac{K \prod_{i=1}^m (s + z_i)}{s \prod_{j=1}^q (s + p_j) \prod_{k=1}^r (s^2 + 2\zeta_k \omega_k s + \omega_k^2)}$$

$$c(t) = a + \sum_{j=1}^q a_j e^{-p_j t} + \sum_{k=1}^r b_k e^{-\zeta_k \omega_k t} \sin(\omega_k \sqrt{1 - \zeta_k^2} t + \varphi)$$

- The poles located far from the $j\omega$ (imaginary) axis have large negative real parts. The exponential terms that correspond to these poles decay very rapidly to zero.
- The poles located nearest the $j\omega$ axis correspond to transient response terms that decay slowly: **dominant poles**

Higher-order systems



Example

- e^{-t} and $e^{-t} \sin 20t$ decay slowly
- e^{-10t} and $e^{-10t} \sin 20t$ decay rapidly

System approximation using the concept of dominant poles

- Consider a system with a transfer function:

$$H(s) = \frac{k(s + z)}{(\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1)(s + p)}$$

- the poles: $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$, $p_3 = -p$.
- Consider the real pole far from the $j\omega$ axis; complex poles are dominant.
- The system order can be reduced by *neglecting the real pole*.
- !!! The gain factor must be multiplied by $1/p$ ($\Rightarrow$ same DC gain).

Dominant poles - Example 1

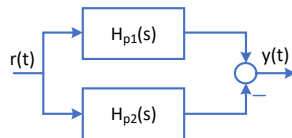
$$H_1(s) = \frac{s+2}{(s^2+2s+2)(s+10)}$$

- Poles: $-1 \pm j$ and -10
- Partial fraction decomposition:

$$H_1(s) = \underbrace{\frac{1}{41} \cdot \frac{4s+9}{s^2+2s+2}}_{H_{p1}(s)} - \underbrace{\frac{1}{41} \cdot \frac{4}{s+10}}_{H_{p2}(s)}$$

$$H_{p1}(s) \xrightarrow{\mathcal{L}^{-1}} \underbrace{Ae^{-t} \sin(t + \varphi_1)}_{\text{decays slowly to 0}}$$

$$H_{p2}(s) \xrightarrow{\mathcal{L}^{-1}} \underbrace{Be^{-10t}}_{\text{decays rapidly to 0}}$$

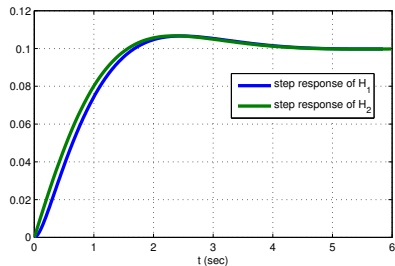


Dominant poles - Example 1

$$H_1(s) = \frac{s + 2}{(s^2 + 2s + 2)(s + 10)}$$

- $p_{1,2} = -1 \pm j$: dominant
- $p_3 = -10$: can be neglected.
- Approximate:

$$H_1(s) = \frac{s + 2}{10(s^2 + 2s + 2)(\frac{1}{10}s + 1)}$$
$$\approx \frac{0.1(s + 2)}{s^2 + 2s + 2} = H_2(s)$$

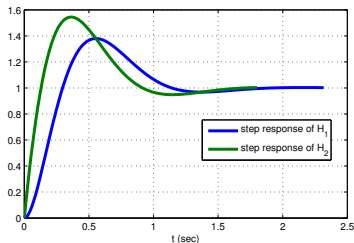


Dominant poles - Example 2

$$H_1(s) = \frac{62.5(s + 2.5)}{(s^2 + 6s + 25)(s + 6.25)}$$

- $p_{1,2} = -3 \pm 4 \cdot j$, $p_3 = -6.25$. Neglect the real pole and obtain:

$$H_2(s) = \frac{10(s + 2.5)}{s^2 + 6s + 25}$$



- $\Rightarrow$ Different step responses !!! (poles are too close to each other)

The Stability of Linear Systems

Stability. Introduction

- Whether a linear system is stable or unstable is a property of the system itself and does not depend on the input.
- **A system is BIBO stable if for any bounded input it has a bounded output (response).**
- Step, sin: bounded. Ramp, impulse: not bounded

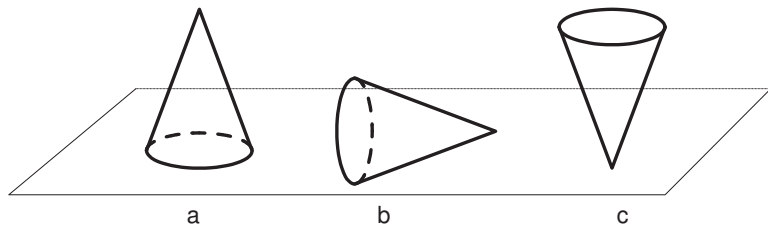
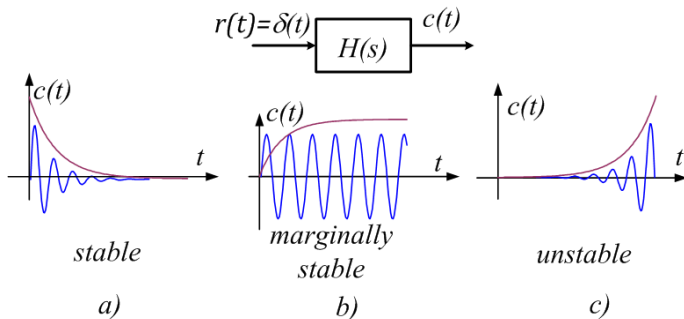


Figure: a. Stable, b. Marginally stable. c. Unstable

System stability

- The **impulse response** can also be used for stability analysis.
- A linear system is stable if and only if the absolute value of its impulse response, integrated over an infinite range, is finite.
- Consequence: the impulse response of a stable system will approach zero as time approaches infinity.



System stability

The system transfer function:

$$H(s) = \frac{C(s)}{R(s)} = \frac{k \prod_{i=1}^m (s + z_i)}{s^n \prod_{j=1}^q (s + \sigma_j) \prod_{k=1}^r (s^2 + 2\alpha_k s + (\alpha_k^2 + \omega_k^2))}$$

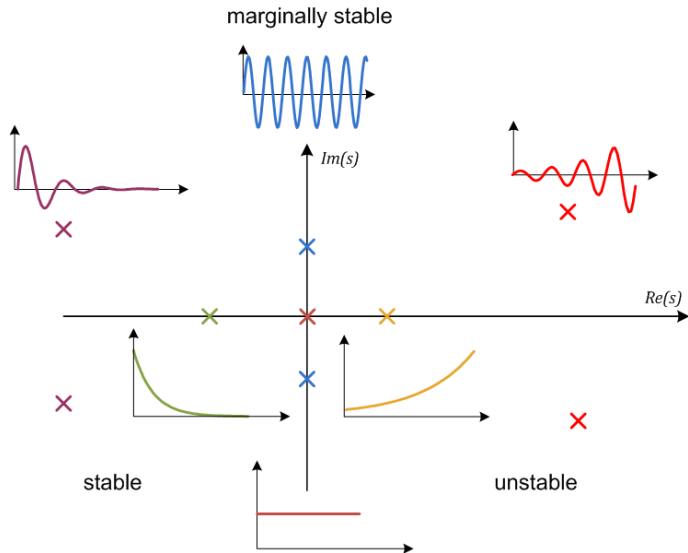
The system poles can be:

- real: $p_j = -\sigma_j$,
- complex conjugates $p_{k1,2} = \alpha_k \pm j\omega_k$,
- poles at the origin of multiplicity n .

The output response for an impulse input is then:

$$c(t) = \sum_{j=1}^q A_j e^{p_j t} + \sum_{k=1}^r B_k \left(\frac{1}{\omega_k} \right) e^{\alpha_k t} \sin \omega_k t$$

System stability-Impulse response



System stability. S-plane criterion

Type of poles and their contribution in the system response

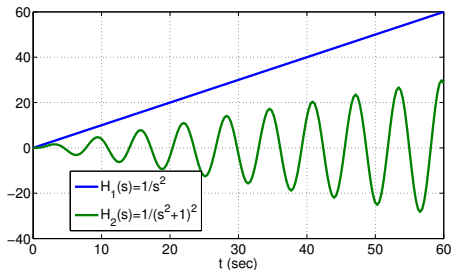
- Real positive poles (p_j) and complex poles with positive real parts (α_k) $\Rightarrow e^{p_j t}$ or $e^{\alpha_k t} \sin \omega_k t$ that increase to ∞
- Real negative poles (p_j) and complex poles with negative real parts (α_k) $\Rightarrow e^{p_j t}$ or $e^{\alpha_k t} \sin \omega_k t$ that approach zero as $t \rightarrow \infty$.
- Complex poles located on the imaginary axis ($\pm j\omega_k$) $\Rightarrow$ undamped sinusoidal term
- One pole at the origin ($p_j = 0$) $\Rightarrow$ a constant term
- Poles at the origin of multiplicity greater than one $n > 1 \Rightarrow At^{n-1}$ which increase towards ∞
- Complex poles located on the imaginary axis of multiplicity greater than one $\Rightarrow At^n \cos(\omega t + \phi)$ which approach infinity as $t \rightarrow \infty$.

System stability. S-plane criterion

Example of systems with poles on the imaginary axis of multiplicity greater than 1.

$$H_1(s) = \frac{1}{s^2}, \quad H_2(s) = \frac{1}{(s^2 + 1)^2}$$

- H_1 has a double pole at the origin and H_2 has two pairs of complex poles at $\pm j$
- System response to impulse:



System stability. S-plane criterion

A necessary and sufficient condition for a system to be **stable** is that **all** the poles of the system transfer function have negative real parts.

- A system is not stable if *not* all the poles are in the left-hand s-plane.
- A *marginally stable or critically stable* system has poles on the imaginary axis, with all the other roots in the left-hand plane.
- An *unstable* system, has at least one root in the right-hand half plane or repeated $j\omega$ (imaginary) roots or multiple poles at the origin.

S-plane criterion. Examples

- Stable system:

$$H_1(s) = \frac{1}{(s+1)(s+2)}$$

all the poles are negative: $p_1 = -1$ and $p_2 = -2$.

- Stable system:

$$H_2(s) = \frac{1}{(s+1)(s^2+2s+2)}$$

all the poles are located in the left half s-plane: $p_1 = -1$, $p_{2,3} = -1 \pm j$

- Marginally stable system:

$$H_3(s) = \frac{1}{s(s+1)}$$

one pole at the origin $p_1 = 0$ and one pole negative $p_2 = -1$.

S-plane criterion. Examples

- Marginally stable system:

$$H_3(s) = \frac{1}{s^2 + 4}$$

a pair of complex poles on the imaginary axis $p_{1,2} = \pm 2j$.

- Unstable system:

$$H_4(s) = \frac{1}{(s+1)(s+2)(s-1)}$$

one pole is positive $p_1 = 1$ and the other poles $p_2 = -1$, $p_3 = -2$ are negative.

- Unstable system:

$$H_5(s) = \frac{1}{s^3(s+1)}$$

a triple pole at the origin $p_{1,2,3} = 0$ and a negative pole $p_4 = -1$.

S-plane criterion. Examples

- Marginally stable system:

$$H_6(s) = \frac{1}{(s^2 + 4)(s^2 + 1)}$$

two *different* pairs of complex poles on the imaginary axis $p_{1,2} = \pm 2j$, $p_{3,4} = \pm j$.

- Unstable system:

$$H_7(s) = \frac{1}{(s^2 + 4)^2}$$

multiple poles on the imaginary axis $p_{1,2} = \pm 2j$, $p_{3,4} = \pm 2j$ (same locations)

Routh-Hurwitz criterion

- The Routh-Hurwitz stability method provides an answer to the question of stability by considering the **characteristic equation** of the system.
- Write the characteristic equation in the form:

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$$

- *Necessary conditions*
 - All the coefficients must have the same sign
 - All the coefficients are nonzero.

The system is not stable if they are not satisfied. *Examples:*

$$H_1(s) = \frac{1}{s^2 + 1}, \quad H_2(s) = \frac{s + 2}{s^2 - 1}, \quad H_3(s) = \frac{s + 3}{s^3 - s^2 + s + 1}$$

Routh-Hurwitz criterion

- *Sufficient condition.* If the necessary conditions are fulfilled, arrange the coefficients of the polynomial in the **Routh array**.

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

Routh array:

$$\begin{array}{lcl} s^n & : & a_n \quad a_{n-2} \quad a_{n-4} \quad a_{n-6} \quad . \quad . \\ s^{n-1} & : & a_{n-1} \quad a_{n-3} \quad a_{n-5} \quad a_{n-7} \quad . \quad . \\ s^{n-2} & : & b_1 \quad b_2 \quad b_3 \quad b_4 \quad . \quad . \\ s^{n-3} & : & c_1 \quad c_2 \quad c_3 \quad c_4 \quad . \quad . \\ . & : & . \quad . \quad . \quad . \quad . \quad . \\ . & : & . \quad . \quad . \quad . \quad . \quad . \\ s^2 & : & e_1 \quad e_2 \\ s^1 & : & f_1 \\ s^0 & : & g_1 \end{array}$$

Routh-Hurwitz criterion

Coefficients $b_1, b_2, \dots$ are computed from the two previous rows.

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	...
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	...
s^{n-2}	b_1	b_2	b_3	...	
s^{n-3}	c_1	c_2	c_3	...	
...					
s^1	f_1	...			
s^0	g_1				

$$b_1 = \frac{- \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-2} - a_na_{n-3}}{a_{n-1}}$$

Routh-Hurwitz criterion

Coefficients $b_1, b_2, \dots$ are computed from the two previous rows.

$$\begin{array}{cccccc} s^n & : & \boxed{a_n} & a_{n-2} & \boxed{a_{n-4}} & a_{n-6} & \dots \\ s^{n-1} & : & \boxed{a_{n-1}} & a_{n-3} & \boxed{a_{n-5}} & a_{n-7} & \dots \\ s^{n-2} & : & b_1 & b_2 & b_3 & \dots & \\ s^{n-3} & : & c_1 & c_2 & c_3 & \dots & \\ \dots & & & & & & \\ s^1 & : & f_1 & \dots & & & \\ s^0 & : & g_1 & & & & \end{array}$$


$$b_2 = \frac{- \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-4} - a_na_{n-5}}{a_{n-1}}$$

Routh-Hurwitz criterion

Coefficients $b_1, b_2, \dots$ are computed from the two previous rows.

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	...
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	...
s^{n-2}	b_1	b_2	b_3	...	
s^{n-3}	c_1	c_2	c_3	...	
...					
s^1	f_1	...			
s^0	g_1				

$$b_3 = \frac{- \begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-6} - a_na_{n-7}}{a_{n-1}}$$

Routh-Hurwitz criterion

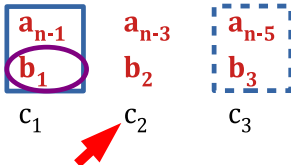
The same pattern is followed in evaluating the c 's, .. e 's, f , g , using the **coefficients of the two previous rows**

$$\begin{array}{cccccc} s^n & : & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \dots \\ s^{n-1} & : & \boxed{a_{n-1}} & \boxed{a_{n-3}} & a_{n-5} & a_{n-7} & \dots \\ s^{n-2} & : & \boxed{b_1} & \boxed{b_2} & b_3 & \dots & \\ s^{n-3} & : & c_1 & c_2 & c_3 & \dots & \\ \dots & & & & & & \\ s^1 & : & f_1 & \dots & & & \\ s^0 & : & g_1 & & & & \end{array}$$


$$c_1 = \frac{- \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

Routh-Hurwitz criterion

The same pattern is followed in evaluating the c 's, e 's, f , g , using the **coefficients of the two previous rows**

$$\begin{array}{cccccc} s^n & : & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \dots \\ s^{n-1} & : & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \dots \\ s^{n-2} & : & b_1 & b_2 & b_3 & \dots & \\ s^{n-3} & : & c_1 & c_2 & c_3 & \dots & \\ \dots & & & & & & \\ s^1 & : & f_1 & \dots & & & \\ s^0 & : & g_1 & & & & \end{array}$$


$$c_2 = \frac{- \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1} = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

Routh-Hurwitz criterion

Routh's stability criterion states that:

The number of roots of the characteristic polynomial $q(s)$ with positive real parts is equal to the number of changes in sign of the coefficients in the first column of the array.

$$\begin{array}{lcl} s^n & : & a_n \quad a_{n-2} \quad a_{n-4} \quad a_{n-6} \quad . \quad . \\ s^{n-1} & : & a_{n-1} \quad a_{n-3} \quad a_{n-5} \quad a_{n-7} \quad . \quad . \\ s^{n-2} & : & b_1 \quad b_2 \quad b_3 \quad b_4 \quad . \quad . \\ . & : & . \quad . \quad . \quad . \quad . \quad . \\ . & : & . \quad . \quad . \quad . \quad . \quad . \\ s^2 & : & e_1 \quad e_2 \quad . \quad . \quad . \quad . \\ s^1 & : & f_1 \quad . \quad . \quad . \quad . \quad . \\ s^0 & : & g_1 \quad . \quad . \quad . \quad . \quad . \end{array}$$

All poles are in the left-half s-plane $\Leftrightarrow$ all terms in the first column of the array have the same sign.

Example

Consider the following polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Check the necessary conditions (all coefficients have the same sign and are non-zero). Routh array:

$$\begin{array}{rcll} s^4 & : & 1 & 3 & 5 \\ s^3 & : & 2 & 4 & 0 \\ s^2 & : & \frac{2 \cdot 3 - 1 \cdot 4}{2} = 1 & \frac{2 \cdot 5 - 1 \cdot 0}{2} = 5 & \\ s^1 & : & \frac{1 \cdot 4 - 2 \cdot 5}{1} = -6 & & \\ s^0 & : & \frac{-6 \cdot 5 - 1 \cdot 0}{-6} = 5 & & \end{array}$$

$\Rightarrow$ Two changes in sign $\Rightarrow$ two roots with positive real parts. Roots:
 $p_1 = 0.2 + 1.4i, p_2 = 0.2 - 1.4i, p_3 = -1.2 + 0.8i, p_4 = -1.2 - 0.8i.$

Example. Special cases

A zero term in the array is replaced by a very small positive number ϵ and the rest of the array is evaluated.

$$s^3 + 2s^2 + s + 2 = 0$$

The necessary conditions are fulfilled (all coefficients have the same sign and are non-zero).

The array of coefficients is:

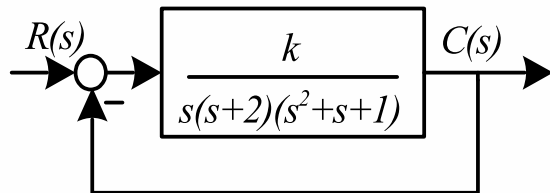
$$\begin{array}{lcl} s^3 & : & 1 \quad 1 \\ s^2 & : & 2 \quad 2 \\ s^1 & : & 0 \approx \epsilon \\ s^0 & : & 2 \end{array}$$

$\Rightarrow$ pair of imaginary roots.

If the sign of the coefficient above zero is opposite that below it, it indicates that there is one sign change.

Example. Closed-loop control system

Determine the range of k for stability.



- The closed-loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{\frac{k}{s(s+2)(s^2+s+1)}}{1 + \frac{k}{s(s+2)(s^2+s+1)}} = \frac{k}{s(s+2)(s^2+s+1) + k}$$

- The characteristic equation is:

$$s^4 + 3s^3 + 3s^2 + 2s + k = 0$$

Example. Closed-loop control system

- *Necessary condition:* $k > 0$. All the other coefficients of the characteristic polynomial are positive and non-zero.
- The array of coefficients:

$$\begin{array}{rcl} s^4 & : & 1 \quad 3 \quad k \\ s^3 & : & 3 \quad 2 \quad 0 \\ s^2 & : & 7/3 \quad k \\ s^1 & : & 2 - 9k/7 \\ s^0 & : & k \end{array}$$

- For stability: all coefficients in the first column must be positive:

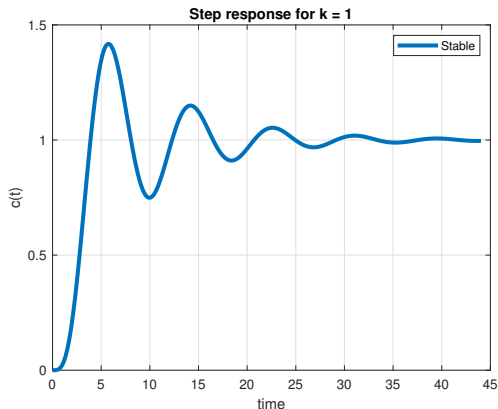
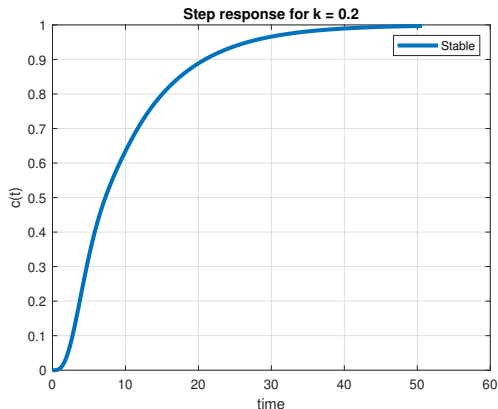
$$k > 0, \text{ and } 2 - \frac{9 \cdot k}{7} > 0, \Rightarrow 0 < k < \frac{14}{9}$$

- When $k = 14/9$, the system is critically stable and the closed-loop system response becomes oscillatory.

Example. Closed-loop control system

Step response for the closed loop system.

- $k = 0.2$: closed-loop system is stable (left)
- $k = 1$: closed-loop system is stable (right)



Example. Closed-loop control system

Step response for the closed loop system.

- $k = \frac{14}{9}$: closed-loop system is critically stable (left)
- $k = 3$: closed-loop system is unstable (right)

