

System analysis. Steady-state error

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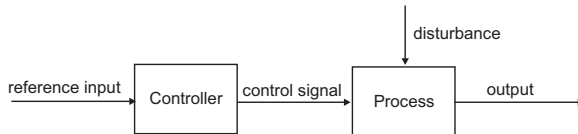
Goals of a control system. Error signal

- There are two main goals of a control system: **stability** and **tracking**.
- **Stability**: we want to guarantee that the system is stable for all possible initial conditions and input signals.
- **Tracking**: we want the **actual system response** to follow a **desired response** of the system; that is, we want the **output** of the system to **track a reference input**.
- The **error signal** is defined as the difference between the desired response of the system, and the actual response.

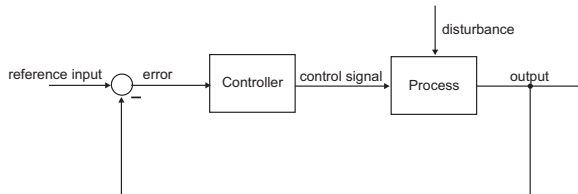
Goals of a control system. Error signal

Error signal = reference input signal - output signal

The error signal is calculated the same for an open-loop control systems



and closed-loop control systems

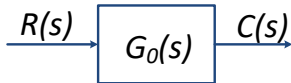


However, notice that the controller of the closed loop system takes explicitly into account the error through feedback. Why does this matter?

Steady-state error

The **steady-state error** = the error between the reference input ($r(t)$) and the output ($c(t)$) at steady-state.

$$e(t) = r(t) - c(t), \quad e_{ss} = \lim_{t \rightarrow \infty} e(t), \quad e_{ss} = \lim_{t \rightarrow \infty} (r(t) - c(t))$$



The steady-state error is calculated only if the system response has a steady state - the system is stable!

Laplace transform of the error: $E(s) = R(s) - C(s) = (1 - G_0(s))R(s)$

The *final value theorem* (FVT) states that:

$$\text{if } \lim_{t \rightarrow \infty} e(t) \text{ exists, then: } e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

Apply the FVT only if all nonzero poles of $E(s)$ have negative real parts and $E(s)$ has no more than one pole at the origin

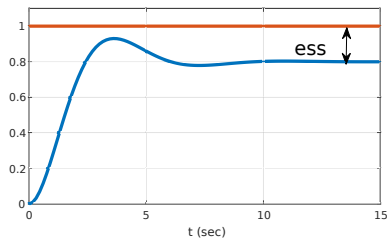
Steady-state error. Example 1

Consider a system with the transfer function: $G(s) = \frac{C(s)}{R(s)} = \frac{0.8}{s^2 + s + 1}$.

Consider the step input $r(t) = 1 \Rightarrow R(s) = \frac{1}{s}$.

The steady-state error can be calculated as:

$$e_{ss} = \lim_{s \rightarrow 0} s(R(s) - C(s)) = \lim_{s \rightarrow 0} s(1 - G(s))R(s) = \lim_{s \rightarrow 0} s\left(1 - \frac{0.8}{s^2 + s + 1}\right)\frac{1}{s} = 0.2.$$



In the figure: input = 1, output at steady-state = 0.8, steady-state error $e_{ss} = 1 - 0.8 = 0.2$.

Steady-state error. Example 2

Consider the transfer function: $G(s) = \frac{C(s)}{R(s)} = \frac{2.5}{s^2 + 3s + 2}$

For the input $r(t) = 2 \Rightarrow R(s) = \frac{2}{s}$. The steady-state error is:

$$e_{ss} = \lim_{s \rightarrow 0} s \left(1 - \frac{2.5}{s^2 + 3s + 2} \right) \frac{2}{s} = \left(1 - \frac{2.5}{2} \right) \cdot 2 = -0.5.$$

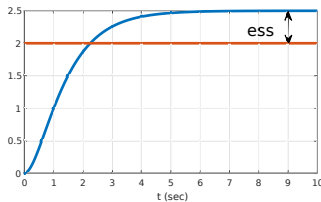


Figure: Steady state error $e_{ss} = 2 - 2.5 = -0.5$.

Steady-state error. Example 2

Consider as application the velocity control of a DC motor for actuating the propellers of a quadcopter drone (ESC - electronics speed controller).

The DC motor, with input voltage and output velocity, has the transfer function:

$$G(s) = \frac{C(s)}{R(s)} = \frac{2.5}{s^2 + 3s + 2}$$

What is the interpretation of the steady state error?

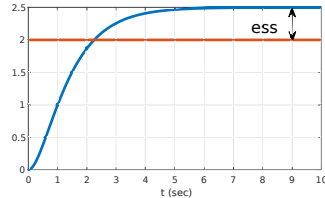


Figure: Steady state error $e_{ss} = 2 - 2.5 = -0.5$ rad/sec.

Steady-state error. Example 3

Consider the transfer function: $G(s) = \frac{C(s)}{R(s)} = \frac{2}{s^2 + s + 2}$

The input is $r(t) = 2 \Rightarrow R(s) = \frac{2}{s}$. The steady-state error:

$$e_{ss} = \lim_{s \rightarrow 0} s \left(1 - \frac{2}{s^2 + s + 2} \right) \frac{2}{s} = \left(1 - \frac{2}{2} \right) \cdot 2 = 0.$$

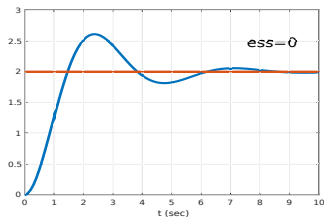
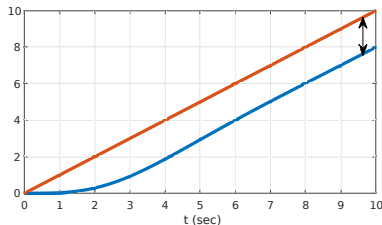


Figure: input = 2, output at steady-state = 2 , steady-state error $e_{ss} = 2 - 2 = 0$.

Steady-state error. Example 4 - ramp input

Consider a system with the transfer function: $G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$ and the input $r(t) = t \Rightarrow R(s) = \frac{1}{s^2}$. The steady-state error:

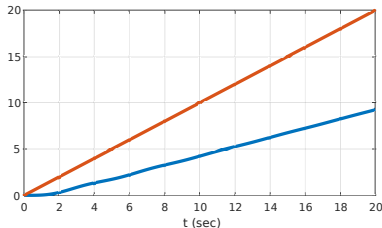
$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} s \left(1 - \frac{1}{s^3 + 2s^2 + 2s + 1} \right) \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{s^3 + 2s^2 + 2s}{s^3 + 2s^2 + 2s + 1} \frac{1}{s} = \\ &= \lim_{s \rightarrow 0} \frac{s^2 + 2s + 2}{s^3 + 2s^2 + 2s + 1} = 2 \end{aligned}$$



Steady-state error. Examples 5 - ramp input

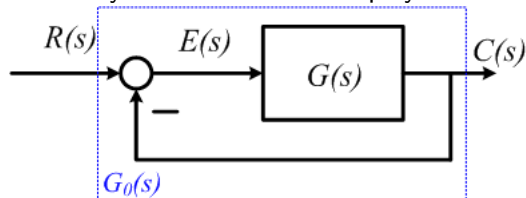
Consider a system with the transfer function: $G(s) = \frac{1}{s^3 + s^2 + 3s + 2}$ and the input $r(t) = t \Rightarrow R(s) = \frac{1}{s^2}$. The steady-state error:

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} s \left(1 - \frac{1}{s^3 + s^2 + 3s + 2} \right) \frac{1}{s^2} = \lim_{s \rightarrow 0} \left(1 - \frac{1}{s^3 + s^2 + 3s + 2} \right) \frac{1}{s} \\ &= \lim_{s \rightarrow 0} \frac{s^3 + s^2 + 3s + 1}{s^3 + s^2 + 3s + 2} \cdot \frac{1}{s} = \infty \end{aligned}$$



Steady-state error for unity feedback systems

For a unity feedback closed-loop system:



The closed-loop system $G_0(s)$ must be stable!

the error is:

$$E(s) = R(s) - C(s) = R(s) - R(s)G_0(s) = R(s)(1 - G_0(s))$$

or

$$E(s) = R(s)\left(1 - \frac{G(s)}{1 + G(s)}\right) = R(s)\frac{1}{1 + G(s)}$$

Final value theorem:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

Steady-state error for unity feedback systems

- using the closed-loop transfer function $G_0(s)$:

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - G_0(s))R(s)$$

- using the open-loop transfer function $G(s)$:

$$e_{ss} = \lim_{s \rightarrow 0} s \left(\frac{1}{1 + G(s)} R(s) \right)$$

For a unit step reference input: $r(t) = 1$ or $R(s) = 1/s$:

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - G_0(s)) \frac{1}{s} = \lim_{s \rightarrow 0} (1 - G_0(s)) = 1 - G_0(0)$$

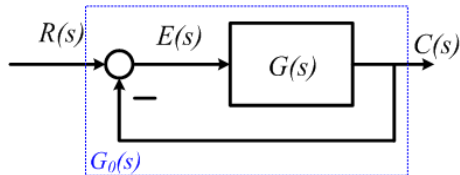
or

$$e_{ss} = \lim_{s \rightarrow 0} s \left(\frac{1}{1 + G(s)} \right) \left(\frac{1}{s} \right) = \frac{1}{1 + G(0)}$$

Steady-state error - Example

Consider closed-loop system with the open-loop transfer function:

$$G(s) = \frac{k}{Ts + 1}$$

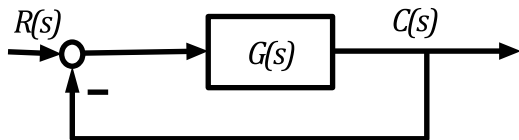


For a step input: $R(s) = 1/s$, the steady-state error is:

$$e_{ss} = \frac{1}{1 + G(0)} = \frac{1}{1 + k}$$

Static error constants

- Consider a closed-loop negative feedback control system



$$G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{s^N \prod_{i=1}^Q (s + p_i)}$$

- $G(s)$ = the open-loop transfer function
- unity feedback
- N = type number
- The steady-state error:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} (r(t) - c(t))$$

$$e_{ss} = \lim_{s \rightarrow 0} s(R(s) - C(s)) = \lim_{s \rightarrow 0} sR(s) \left(1 - \frac{G(s)}{1 + G(s)}\right) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

Steady-state error

Step input: $r(t) = 1 \Rightarrow R(s) = \frac{1}{s}$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s} \frac{1}{1 + G(s)} = \frac{1}{1 + G(0)} = \frac{1}{1 + K_p}$$

$$K_p = \lim_{s \rightarrow 0} G(s) = \text{the position error constant}$$

■ $N = 0$:

$$G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{\prod_{i=1}^Q (s + p_i)} \Rightarrow K_p = \text{const} \Rightarrow e_{ss} = \text{const}$$

■ $N \geq 1$:

$$G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{s^N \prod_{i=1}^Q (s + p_i)} \Rightarrow K_p = \infty \Rightarrow e_{ss} = 0$$

Steady-state error

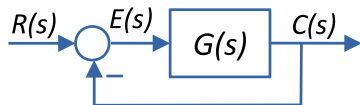
Ramp input: $r(t) = t \Rightarrow R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s^2} \frac{1}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{s(1 + G(s))} = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{K_v}$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = \text{the velocity error constant}$$

- $N = 0$: $G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{\prod_{i=1}^Q (s + p_i)} \Rightarrow K_v = 0 \Rightarrow e_{ss} = \infty$
- $N = 1$: $G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{s \prod_{i=1}^Q (s + p_i)} \Rightarrow K_v = \text{const} \Rightarrow e_{ss} = \text{const}$
- $N \geq 2$: $G(s) = \frac{k \prod_{i=1}^M (s + z_i)}{s^N \prod_{i=1}^Q (s + p_i)} \Rightarrow K_v = \infty \Rightarrow e_{ss} = 0$

Example 1



$$G(s) = \frac{s + 4}{(s + 1)(s + 2)}$$

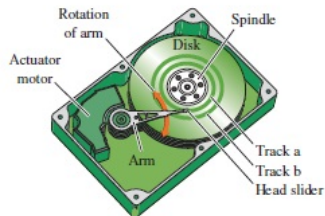
- Check the stability of the closed-loop system.
- The position error constant and the steady-state error for a step input:

$$K_p = \lim_{s \rightarrow 0} G(s) = \lim_{s \rightarrow 0} \frac{s + 4}{(s + 1)(s + 2)} = 2, \quad e_{ss} = \frac{1}{1 + K_p} = \frac{1}{3}$$

- The velocity error constant and the steady-state error for a ramp input:

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \cdot \frac{s + 4}{(s + 1)(s + 2)} = 0, \quad e_{ss} = \frac{1}{K_v} = \infty$$

Example 2 - hard disk drive



$$G(s) = \frac{4}{s(s+2)}$$

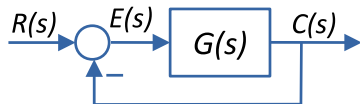
- Check the stability of the closed-loop system.
- The position error constant and the steady-state error for a step input:

$$K_p = \lim_{s \rightarrow 0} G(s) = \lim_{s \rightarrow 0} \frac{4}{s(s+2)} = \infty, \quad e_{ss} = \frac{1}{1 + K_p} = 0$$

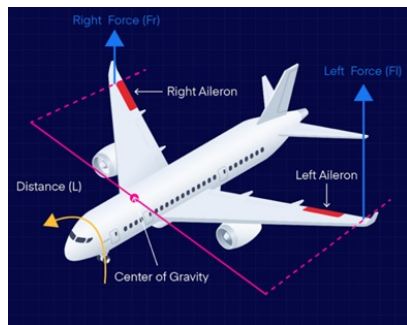
- The velocity error constant and the steady-state error for a ramp input:

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \cdot \frac{4}{s(s+2)} = 2, \quad e_{ss} = \frac{1}{K_v} = \frac{1}{2}$$

Example 3 - aileron of an aircraft



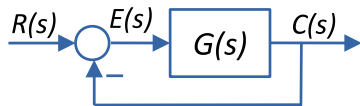
$$G(s) = \frac{50}{(s-1)(s^2 + 10s + 40)}$$



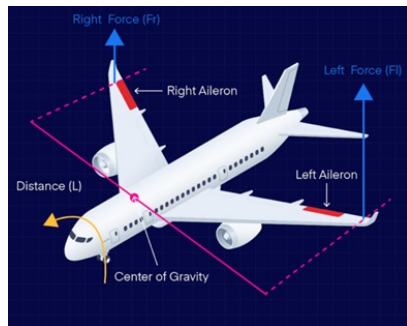
- The output C is the lateral position of the aircraft, while the input R the aileron angle
- The open-loop system $G(s)$ - unstable, but the closed-loop system

$$G_0(s) = \frac{G(s)}{1 + G(s)} = \frac{50}{s^3 + 9s^2 + 30s + 10} \text{ is stable.}$$

Example 3 - aileron of an aircraft



$$G(s) = \frac{50}{(s-1)(s^2+10s+40)}$$



- The position error constant and the steady-state error for a step input:

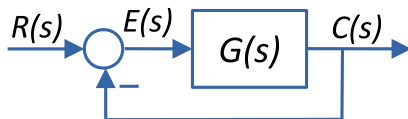
$$K_p = \lim_{s \rightarrow 0} G(s) = \lim_{s \rightarrow 0} \frac{50}{(s-1)(s^2+10s+40)} = -\frac{5}{4}, \quad e_{ss} = \frac{1}{1+K_p} = -4$$

- The velocity error constant and the steady-state error for a ramp input:

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \cdot \frac{50}{(s-1)(s^2+10s+40)} = 0, \quad e_{ss} = \infty$$

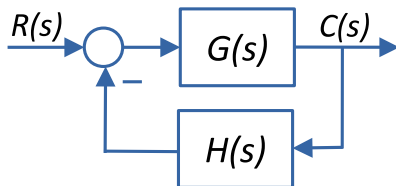
Exercise

Show that for a negative unity feedback closed-loop system:



- The steady-state error for a step input is 0 if $G(s)$ has a pole at the origin (type-1 system)
- The steady-state error for a ramp input is 0 if $G(s)$ has two poles at the origin (type-2 system)

The steady-state error for non-unity feedback closed-loop systems

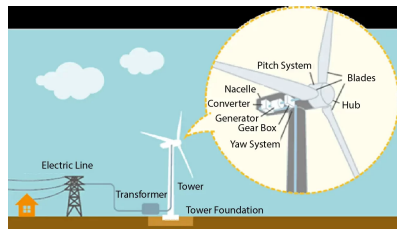
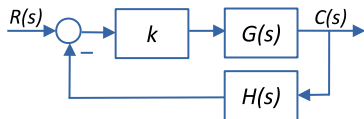


- The transfer function of the closed-loop system $G_0(s) = \frac{G(s)}{1 + G(s)H(s)}$ - stable
- The steady-state error:

$$e_{ss} = \lim_{t \rightarrow \infty} (r(t) - c(t)) = \lim_{s \rightarrow 0} s(R(s) - C(s)) = \lim_{s \rightarrow 0} s \cdot R(s) \cdot (1 - G_0(s))$$

Example - wind turbine

Consider the wind turbine generator speed control system in the absence of wind disturbances. The input is the desired generator speed R , while the output is the actual generator speed C .



Turbine and generator simplified model (Dorf&Bishop2022):

$$G(s) = \frac{7000}{5s + 1},$$

and speed sensor model

$$H(s) = -\frac{1}{0.1s + 1}.$$

Example - wind turbine

Determine the value of the parameter k ($k > 0$) so that the steady-state error is 0 for a step input $r(t) = 4$, $t > 0$.

- Closed-loop:

$$G_0(s) = \frac{kG(s)}{1 + kG(s)H(s)} = \frac{7000k(0.1s + 1)}{0.5s^2 + 5.1s + (1 - 7000k)}$$

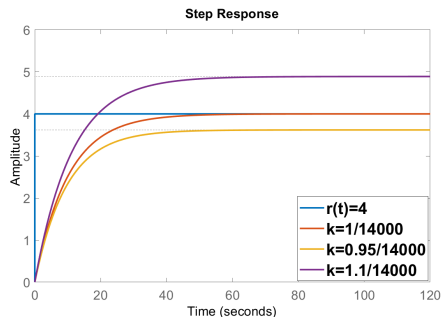
- Check the stability of $G_0(s)$: for example using the RH criterion for the characteristic polynomial $0.5s^2 + 5.1s + (1 - 7000k)$.

Example - wind turbine

■ Steady-state error:

$$e_{ss} = \lim_{s \rightarrow 0} s \cdot R(s) \cdot (1 - G_0(s)) = \lim_{s \rightarrow 0} s \cdot \frac{4}{s} \cdot \left(1 - \frac{7000k(0.1s + 1)}{0.5s^2 + 5.1s + (1 - 7000k)} \right)$$

$$e_{ss} = 4 \cdot (1 - G_0(0)) = 4 \cdot \left(1 - \frac{7000k}{1 - 7000k} \right) = 0 \quad \Rightarrow \quad k = \frac{1}{14000}$$

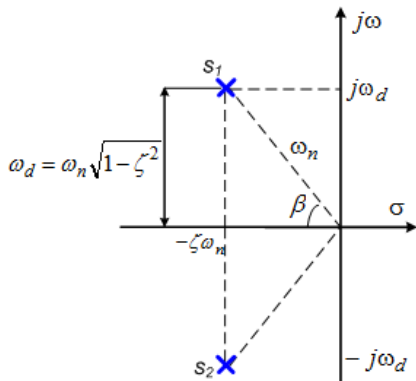


System analysis. Applications

Some notes on the location of the second-order system poles and the characteristics of transient response

Technical University of Cluj-Napoca

Complex poles of a second-order system



$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$\operatorname{Re}(s_{1,2}) = -\zeta\omega_n, \quad \operatorname{Im}(s_{1,2}) = \omega_d = \omega_n\sqrt{1-\zeta^2}$$

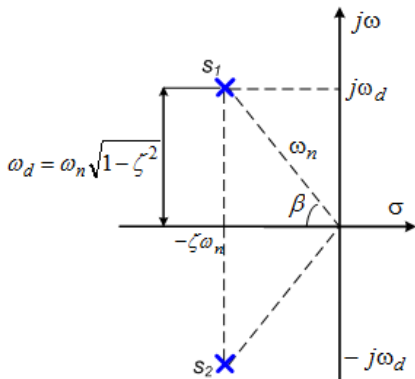
■ settling time:

$$t_s = \frac{4}{\zeta\omega_n}$$

■ peak time:

$$t_p = \frac{\pi}{\omega_d}$$

Complex poles of a second-order system



$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$|s_{1,2}| = \sqrt{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)} = \omega_n$$

■ damping factor

$$\cos\beta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

$$\beta = \arccos\zeta$$

■ rise time:

$$t_r = \frac{\pi - \beta}{\omega_d}$$

Example

Consider a general second-order system with the transfer function:

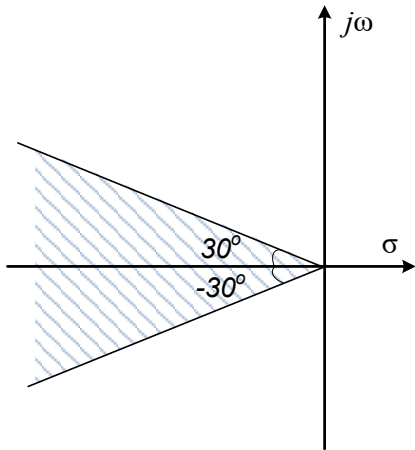
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Show the area where the poles can be located in the complex plane, if the specifications for a step input are:

- (a) Damping factor $\zeta \geq \frac{\sqrt{3}}{2}$.
- (b) Damping factor $\zeta \geq \frac{\sqrt{2}}{2}$ and settling time $t_s < 4$ sec.
- (c) Damping factor $\zeta \leq \frac{\sqrt{2}}{2}$, settling time $t_s < 4$ sec and peak time $t_p = \frac{\pi}{3}$ sec.

Example - (a)

(a) Location of the second-order system poles so that $\zeta \geq \frac{\sqrt{3}}{2}$



■ damping factor

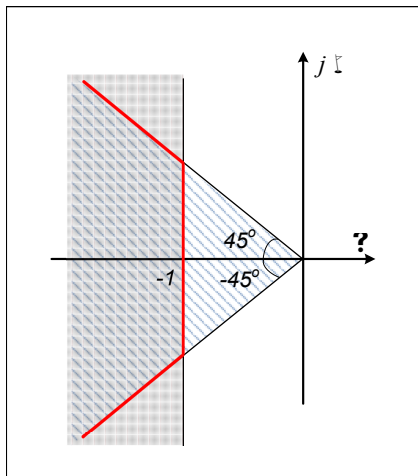
$$\zeta = \cos\beta \Rightarrow \beta = \arccos\zeta$$

$$\zeta \geq \frac{\sqrt{3}}{2} \Rightarrow \cos\beta \geq \frac{\sqrt{3}}{2}$$

$$\Rightarrow \beta \leq \arccos\frac{\sqrt{3}}{2} \Rightarrow \beta \leq 30^\circ$$

Example - (b)

(b) Location of the second-order system poles so that $\zeta \geq \frac{\sqrt{2}}{2}$ and $t_s < 4 \text{ sec}$



■ damping factor

$$\zeta = \cos\beta \geq \frac{\sqrt{2}}{2} \Rightarrow \beta \leq \arccos\frac{\sqrt{2}}{2} = 45^\circ$$

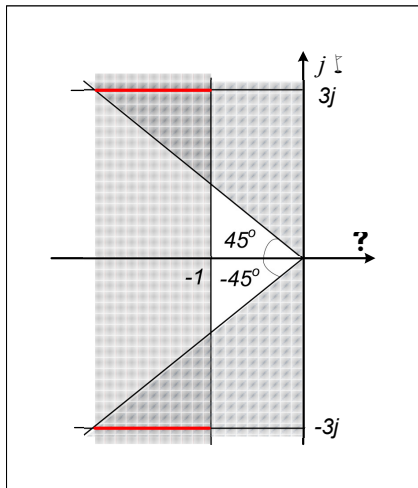
■ settling time

$$t_s = \frac{4}{\zeta\omega_n} < 4 \Rightarrow \zeta\omega_n > 1 \Rightarrow -\zeta\omega_n < -1$$

$$\Rightarrow \text{Re}(\text{pole}) < -1$$

Example - (c)

(c) Location of the second-order system poles so that $\zeta \leq \frac{\sqrt{2}}{2}$, $t_s < 4 \text{ sec}$, $t_p = \frac{\pi}{3} \text{ sec}$.



■ damping factor

$$\zeta = \cos\beta \leq \frac{\sqrt{2}}{2} \Rightarrow \beta \geq \arccos\frac{\sqrt{2}}{2} = 45^\circ$$

■ settling time

$$t_s = \frac{4}{\zeta\omega_n} < 4 \Rightarrow \zeta\omega_n > 1 \Rightarrow -\zeta\omega_n < -1$$

■ peak time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} \Rightarrow \omega_d = 3$$

$$\Rightarrow \text{Re}(\text{pole}) < -1, \text{Im}(\text{pole}) = 3$$