# Root locus analysis

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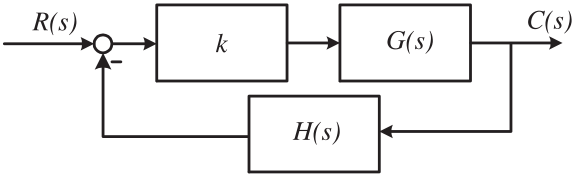
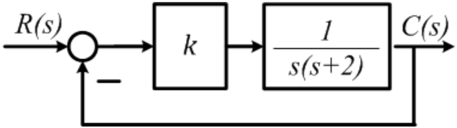
# Introduction

- The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles.
- If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the gain chosen.
- It is important to know how the closed-loop poles move in the s-plane as the loop gain is varied.



# Example




- Open-loop transfer function:

$$kG(s) = k\frac{1}{s(s+2)}$$

  Open-loop poles: 0, $-2$, do not depend on $k$
- Closed-loop transfer function:

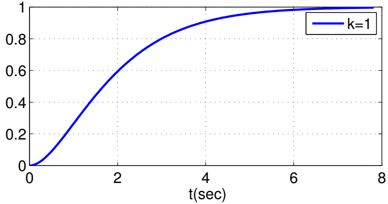
$$G_0(s) = \frac{kG(s)}{1+kG(s)} = \frac{k\frac{1}{s(s+2)}}{1+k\frac{1}{s(s+2)}} = \frac{k}{s^2+2s+k}$$

  Closed-loop poles: depend on $k$

# Example

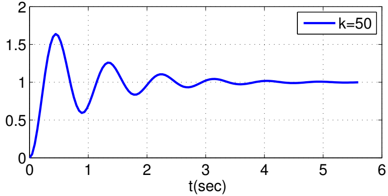
$$k = 1, \quad G_0(s) = \frac{1}{s^2 + 2s + 1}$$

- Poles $s_1 = s_2 = -1$
- The step response is critically damped.



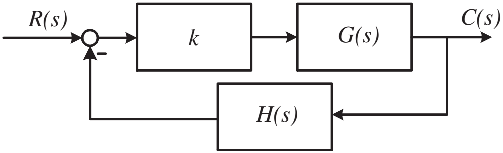

$$k = 50, \quad G_0(s) = \frac{50}{s^2 + 2s + 50}$$

- Poles: $s_{1,2} = -1 \pm 7j$,
- The step response is underdamped.



# Root locus

The root locus is a plot of the roots of the characteristic equation of a **closed-loop system** for all values of a variable parameter in the system, $k \in [0, \infty)$.




- Open-loop transfer function

$$H_d(s) = kG(s)H(s)$$

- Closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{kG(s)}{1 + kG(s)H(s)}$$

# Root locus

- The *characteristic equation* for the closed-loop system:

$$1 + kG(s)H(s) = 0 \quad \Rightarrow \quad kG(s)H(s) = -1$$

- Idea: the values of $s$ that make $kG(s)H(s) = -1$ must satisfy the characteristic equation of the system.
- $kG(s)H(s)$ is a ratio of polynomials in $s$ and $kG(s)H(s)$ is a complex quantity:

$$\mid kG(s)H(s) \mid \angle kG(s)H(s) = -1 + j0$$

- *Angle condition:*

$$\angle kG(s)H(s) = \angle -1 + j0 = \pm 180^o(2q+1), \quad q = 0, 1, 2, \ldots$$

- *Magnitude condition:*

$$|kG(s)H(s)| = |-1 + j0| = 1$$

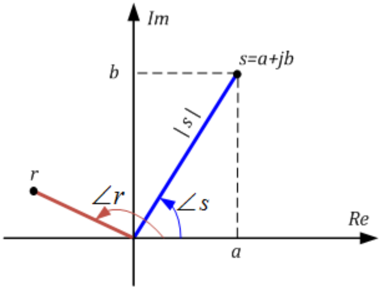
# Remember complex numbers

- The absolute value (or *magnitude* or modulus):

$$|s| = |a + jb| = \sqrt{a^2 + b^2}$$

- The *phase* (or angle, or argument):

$$\angle s = \arg(s) = \arctan \frac{b}{a}, \quad \text{if} \;\; a > 0$$




The phase is measured counterclockwise from the positive real axis.

# Remember complex numbers

- The product of two complex numbers: $s_1 = a + jb$ and $s_2 = c + jd$ has the magnitude:

$$|s_1 s_2| = |s_1| \cdot |s_2| = \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$$

and the phase:

$$\angle(s_1 s_2) = \angle s_1 + \angle s_2 = \angle(a + jb) + \angle(c + jd)$$

- The ratio of two complex numbers has the magnitude:

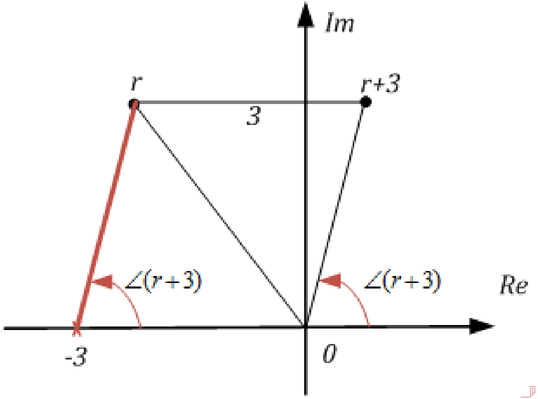
$$\left|\frac{s_1}{s_2}\right| = \frac{|s_1|}{|s_2|} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

and the phase:

$$\angle\frac{s_1}{s_2} = \angle s_1 - \angle s_2 = \angle(a + jb) - \angle(c + jd)$$

# Remember complex numbers

- Consider a complex number $r$. We want to compute the phase of $r+3$.
- The points $r$, $r+3$, 0 and $-3$ form a parallelogram.
- The phase of $r+3$ = the angle (measured counterclockwise) from the positive real axis to the line that connects $r$ and the root of $r+3$, i.e. $-3$.



# Example

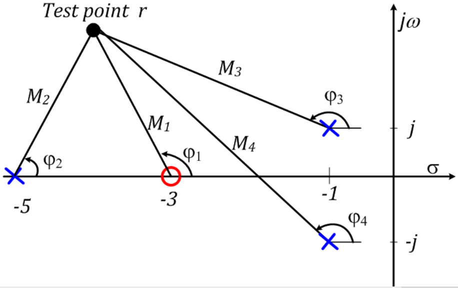
- Consider a closed-loop system with the open-loop transfer function :

$$kG(s)H(s) = \frac{k(s+3)}{(s+5)(s^2+2s+2)}$$

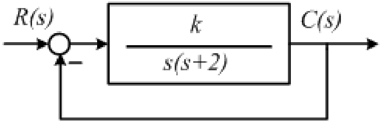
- For a testpoint $s = r$, the angle of $kG(s)H(s)$ is:

$$\angle kG(s)H(s)|_{s=r} = \angle k + \angle(r+3) - \angle(r+5) - \angle(r+1-j) - \angle(r+1+j)$$

$$\angle kG(s)H(s)|_{s=r} = 0 + \varphi_1 - \varphi_2 - \varphi_3 - \varphi_4$$



# Example. Root locus for a second-order system




- The open-loop transfer function $G(s)$:

$$G(s) = \frac{k}{s(s+2)}$$

- The closed-loop transfer function:

$$G_0(s) = \frac{G(s)}{1+G(s)} = \frac{k}{s^2+2s+k}$$

- The characteristic equation is:

$$1 + \frac{k}{s(s+2)} = 0 \quad \text{or } s^2 + 2s + k = 0$$

- Find the locus of the roots of this system for $k \in [0, \infty)$.

# Example. Root locus for a second-order system

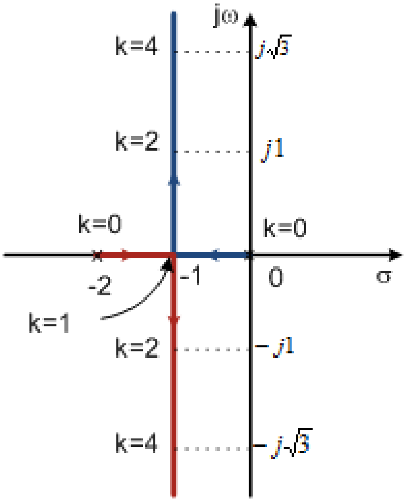
- The roots of the characteristic equation (closed-loop poles):

$$s_1 = -1 + \sqrt{1-k}, \quad s_2 = -1 - \sqrt{1-k}$$

- The roots are real for $k \leq 1$ and complex for $k > 1$.
- For $k = 0$, the closed-loop poles = the open-loop poles: $s_1 = 0$, $s_2 = -2$.
- As $k$ increases in the interval $(0, 1)$, the closed-loop poles move towards the point $(-1, 0)$. Real poles: the system response in overdamped.
- For $k = 1$, the closed-loop poles: $s_1 = s_2 = -1$. The system is critically damped.

# Example. Root locus for a second-order system

- $k \in (1, \infty)$: the closed-loop poles break away from the real axis and become complex.
- Complex poles: $s_{1,2} = -1 \pm \sqrt{k-1}j$.
- The poles move along the vertical line with $Re(s) = -1$, symmetrically
- $k > 1$: underdamped system, oscillatory response.



# Root locus - second order system

Check the angle condition:

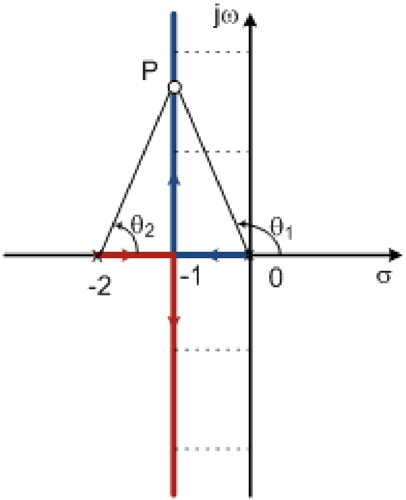
$$\angle \frac{k}{s(s+2)}\Big|_{s=P} =$$
$$= \angle k - \angle s - \angle(s+2) =$$
$$= 0 - \theta_1 - \theta_2 = -180^o$$

because:

for a test point $s = P \in LR$:

$\theta_1 = \angle s$ and $\theta_2 = \angle(s+2)$ and

$$\theta_1 + \theta_2 = 180^o$$



# Root locus - second order system

For a pair of closed-loop poles: $s_{1,2} = -1 \pm j1$, the gain $k$ is determined from the magnitude condition :

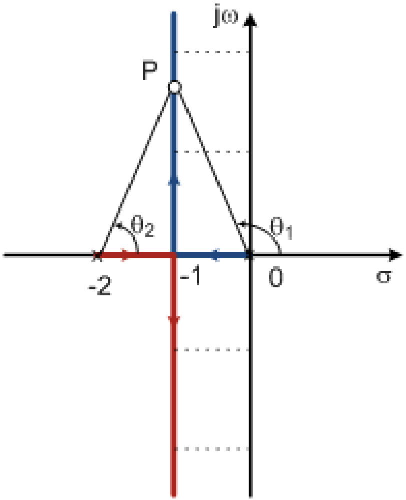
$$|G(s)H(s)| = \left|\frac{k}{s(s+2)}\right|_{s_{1,2}} = 1$$

or

$$k = |s(s+2)|_{s=-1+j1}$$

$$k = |-1+j1| \cdot |1+j1|$$

$$k = \sqrt{1^1+1^2}\sqrt{1^2+1^2} = 2$$



# Root locus procedure

- Write the characteristic equation in the form:

$$1 + kP(s) = 0.$$

- Factor $P(s)$ in terms of $n_p$ poles and $n_z$ zeros

$$1 + k\frac{\prod_{i=1}^{n_z}(s+z_i)}{\prod_{j=1}^{n_p}(s+p_j)} = 0$$

- Locate the open-loop poles and zeros in the s-plane: **x** - poles, **o** - zeros.
- Determine the number of separate loci $SL$. $SL = n_p$, when $n_p \geq n_z$, $n_p =$ number of open-loop poles, $n_z =$ number of open-loop zeros.

# Root locus procedure

- Determine the number of separate loci $SL$. $SL = n_p$.
- Locate the segments of the real axis that are root loci:
  1. Locus lies to the left of an odd number of poles and zeros
  2. Locus begins at a pole and ends at a zero (or infinity)
- The root loci are symmetrical with respect to the real axis.
- The loci proceed to the zeros at infinity along asymptotes centered at $\sigma_A$ and with angles $\Phi_A$.

$$\sigma_A = \frac{\sum(poles) - \sum(zeros)}{n_p - n_z}, \ \Phi_A = \frac{2q+1}{n_p - n_z} \cdot 180^o, \ {}_{q=0,1,\ldots(n_p - n_z - 1)}$$

- From Routh-Hurwitz criterion $\Rightarrow$ the intersection with the imaginary axis (if it does so).

# Root locus procedure

- Determine the breakaway point on the real axis (if any)
  1. Set $k = -\frac{1}{P(s)} = p(s)$, (from $1 + kP(s) = 0$)
  2. Obtain $dp(s)/ds = 0$
  3. Determine roots of (b) or use graphical method to find maximum of $p(s)$.
- Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion

$$\angle P(s) = \pm 180^o(2q+1),\ at\ s = p_j\ or\ z_i.$$

- If required, determine the root locations that satisfy the phase criterion

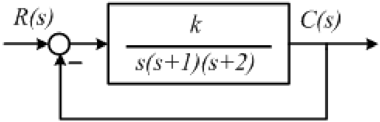
$$\angle P(s) = \pm 180^o(2q+1) \text{ at a root location } s_x$$

- If required, determine the gain $k_x$ at a specific root $s_x$

$$k_x = \frac{\prod_{j=1}^{n_p} \mid s + p_j \mid}{\prod_{i=1}^{n_z} \mid s + z_i \mid} \Big|_{s=s_x}$$

# Root locus. Example

Sketch the root-locus plot and determine the value of $k$ so that the damping factor $\zeta$ of a pair of dominant complex-conjugate closed-loop poles is 0.5 for the closed-loop system:



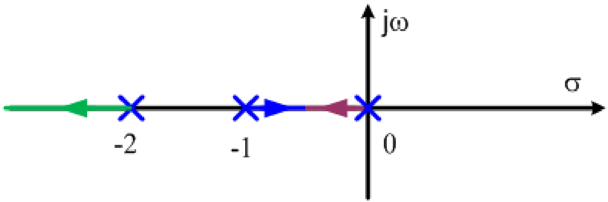

- The characteristic equation:

$$1 + kG(s) = 0, \text{ or } 1 + \frac{k}{s(s+1)(s+2)} = 0$$

- The open-loop transfer function has no zeros: $n_z = 0$, and three poles, $p_1 = 0$, $p_2 = -1$, $p_3 = -2$: $n_p = 3$.

# Root locus. Example

- Locate the open-loop poles of the open-loop transfer function with symbol $\times$.
- Locus has $n_p = 3$ branches.
- Locate the segments of the real axis that are root loci: between $p_1 = 0$ and $p_2 = -1$, and from $p_3 = -2$ to $-\infty$.
- As the locus begins at pole and ends at a zero (or infinity), between 0 and $-1$ we must find a breakaway point.




Open-loop poles and RL on the real axis

# Root locus. Example

- The root locus is symmetrical with respect to the horizontal real axis.
- The locus proceed to the zeros at infinity along asymptotes centered at $\sigma_A$ and with angles $\Phi_A$.

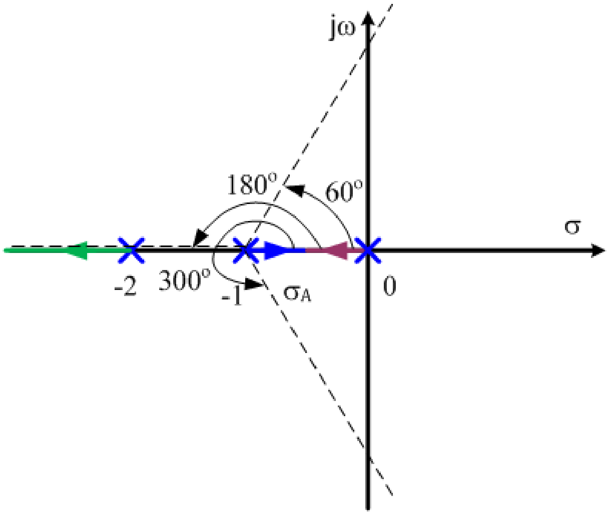
$$\sigma_A = \frac{\sum(p_j) - \sum(z_i)}{n_p - n_z} = \frac{0 - 1 - 2}{3} = -1$$

$$\Phi_A = \frac{2q+1}{n_p - n_z} \cdot 180^o = \frac{2q+1}{3} \cdot 180^o, \quad q = 0, 1, 2$$

or

$$\Phi_A = 60^o, 180^o, 300^o$$

# Root locus. Example




Asymptotes

# Root locus. Example

Routh-Hurwitz criterion:

$$s^3 + 3s^2 + 2s + k = 0$$

Routh array:

$$\begin{array}{lccc} s^3 & : & 1 & 2 \\ s^2 & : & 3 & k \\ s^1 & : & (6-k)/3 & \\ s^0 & : & k & \end{array}$$

$\Rightarrow$ $k = 6$ (system is critically stable $\Rightarrow$ roots on the imaginary axis). Replace $k = 6$ into the characteristic equation:

$$s^3 + 3s^2 + 2s + 6 = 0 \quad \text{or} \quad (s+3)(s^2+2) = 0$$

$$\underline{s_{1,2} = \pm j\sqrt{2}}, \quad s_3 = -3$$

The breakaway point.
From the characteristic equation:

$$1+\frac{k}{s(s+1)(s+2)}=0, \quad \Rightarrow \quad k=-s(s+1)(s+2)=p(s)$$

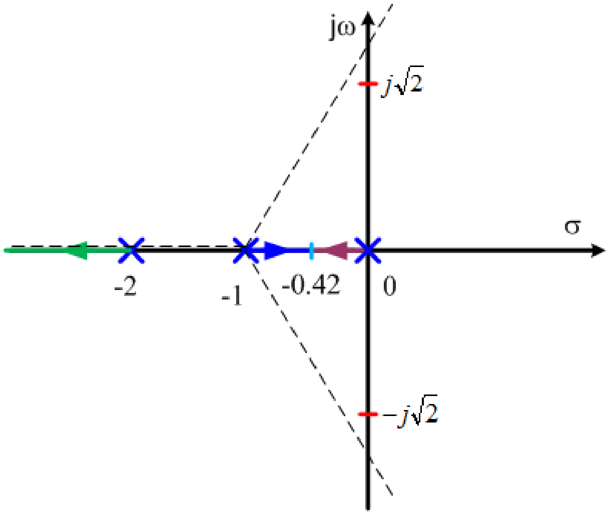
Obtain the solution of the equation:

$$p'(s)=0 \quad \Rightarrow \quad \frac{dp(s)}{ds}=-\frac{d}{ds}\left(s^3+3s^2+2s\right)=0$$

$$\frac{dp(s)}{ds}=-(3s^2+6s+2)=0 \quad \Rightarrow \quad \underline{s_1=-0.4226}, \quad s_2=-1.5774$$

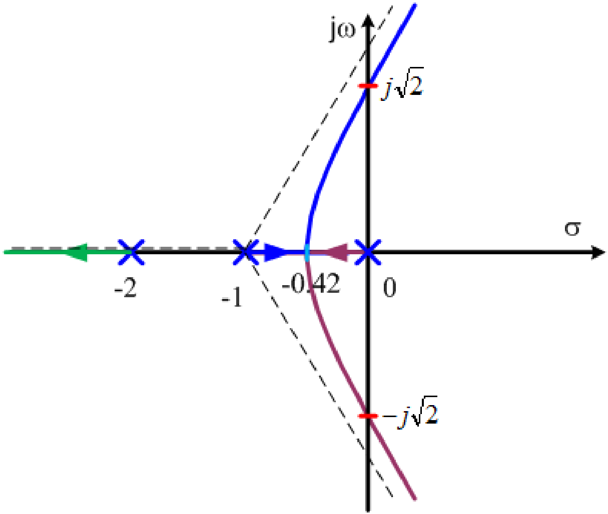
The breakaway point must lie between 0 and -1, $\Rightarrow$ $s_1=-0.4226$

# Root locus. Example




Intersection with the imaginary axis and the breakaway point

# Root locus. Example




Root-locus plot

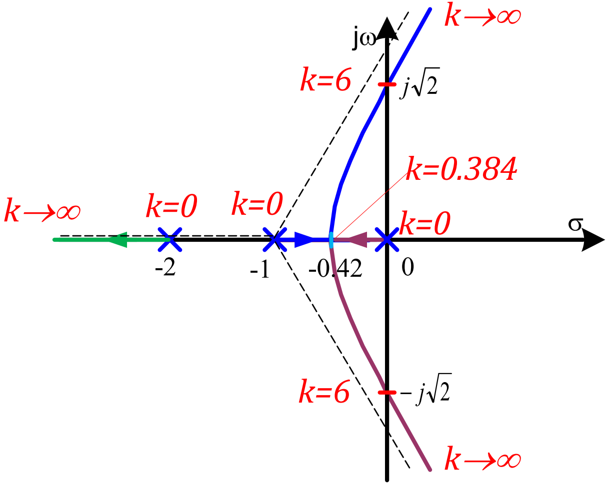
# Root locus. Example. Analysis

The value of $k$ at the breakaway point is obtained from the magnitude condition:

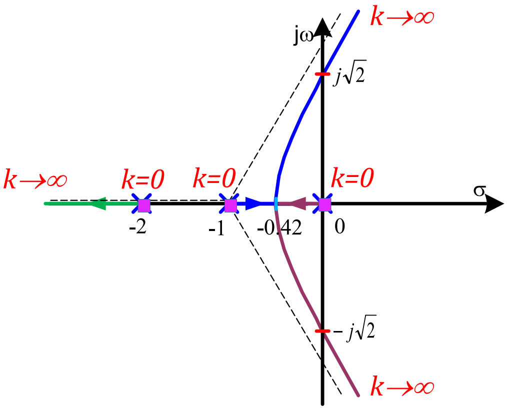
$$|kG(s)|_{s=-0.42} = 1$$

$$\left|\frac{k}{s(s+1)(s+2)}\right|_{s=-0.42} = 1$$

$$k = |-0.42(-0.42+1)(-0.42+2)| = 0.384$$

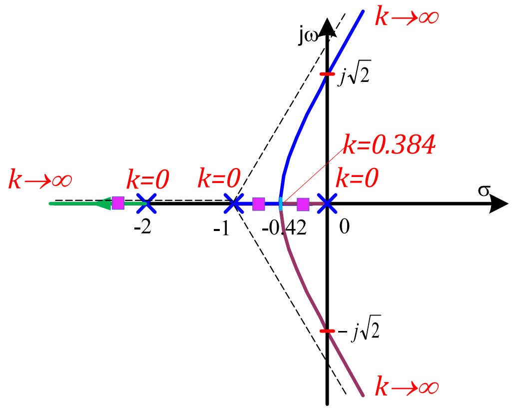


# Root locus. Example. Analysis



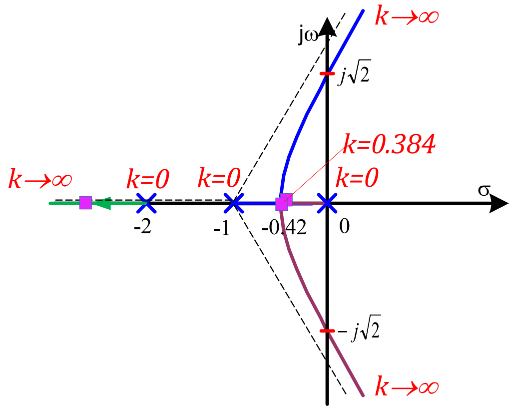

- $k=0$, closed-loop poles ■ = open-loop poles ×
- 2 negative poles, 1 pole at 0 ⇒ closed-loop marginally stable

# Root locus. Example. Analysis



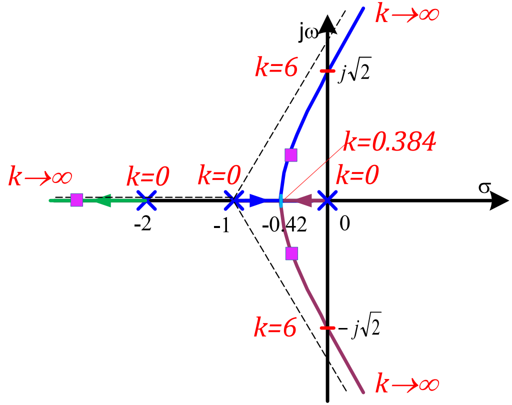

- $k \in (0,\ 0.384)$, closed-loop poles ■
- 3 real negative closed-loop poles ⇒ closed-loop stable, overdamped

# Root locus. Example. Analysis



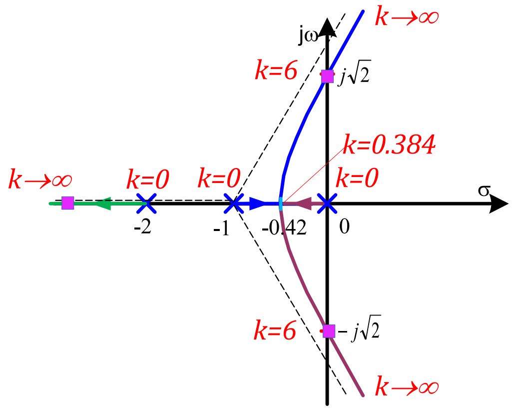

- $k = 0.384$, closed-loop poles
- two real and equal negative poles $(-0.42)$, one negative pole $\Rightarrow$ closed-loop stable, critically damped

# Root locus. Example. Analysis



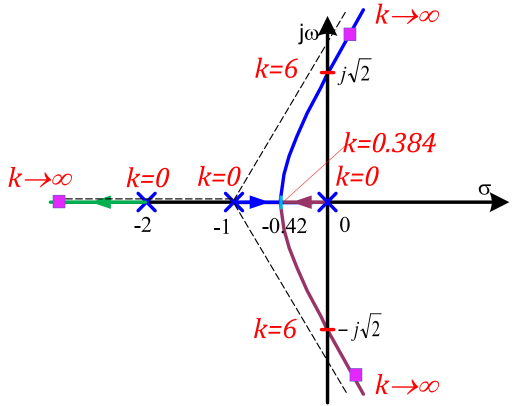

- $k \in (0.384,\ 6)$, closed-loop poles ■
- one negative real pole and two complex poles with negative real part, ⇒ closed-loop stable, underdamped

# Root locus. Example. Analysis




- $k = 6$, closed-loop poles ■
- one negative real pole and two imaginary poles ⇒ closed-loop marginally stable

# Root locus. Example. Analysis



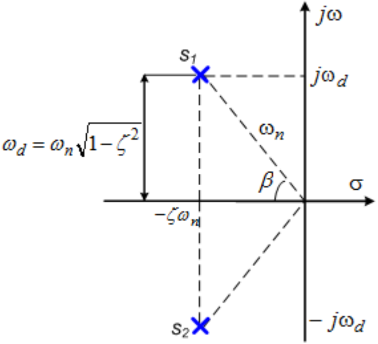

- $k > 6$, closed-loop poles ■
- one negative real pole and two complex poles with positive real parts ⇒ closed-loop unstable

# Root locus. Example. Analysis - summary

- $k = 0$, one pole at the origin, system is critically stable
- $k \in (0, 0.384)$, negative real poles on the left half s-plane, system is stable and overdamped
- $k = 0.384$, two real and equal negative poles, one negative pole, system is critically damped
- $k \in (0.384, 6)$, one negative real pole and two complex poles with negative real part, system is stable and underdamped
- $k = 6$, one negative real pole and two imaginary poles, system is critically stable
- $k > 6$, one negative real pole and two complex poles with positive real parts, system is unstable

# Root locus. Example

*Remember.* The complex poles of a second-order transfer function:




$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$cos\beta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

$$\beta = arccos\zeta$$

Closed-loop poles with $\zeta = 0.5$ lie on the lines passing through the origin and making the angles $\pm arccos\zeta = \pm arccos0.5 = \pm 60^o$ with the negative real axis.

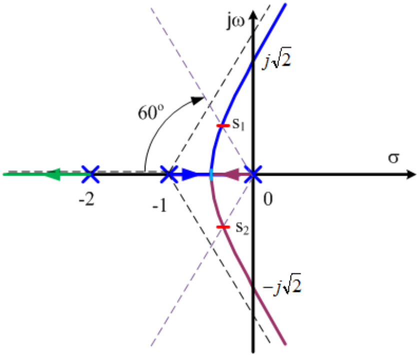



Figure: Root-locus plot and the poles with $\zeta = 0.5$

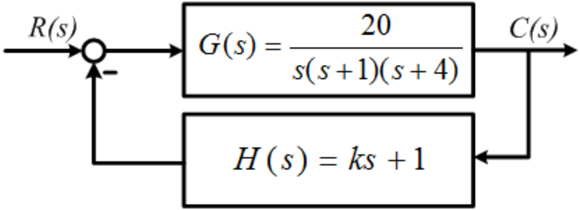
$$s_{1,2} = -0.3337 \pm j0.5780$$

The value of $k$ for $s_{1,2}$ is found from the magnitude condition:

$$k = |s(s+1)(s+2)|_{s_{1,2}} = 1.0383$$

The third pole is found at $s = -2.3326$.
These accurate values can be determined only using a computer.

**Root locus for any variable parameter. Example**




$$G(s) = \frac{20}{s(s+1)(s+4)}, \quad H(s) = 1 + ks$$

The open-loop transfer function is then:

$$G(s)H(s) = \frac{20(1+ks)}{s(s+1)(s+4)}$$

# Root locus. Example

- The characteristic equation:

$$1 + G(s)H(s) = 1 + \frac{20(1+ks)}{s(s+1)(s+4)} = 0$$

$$s^3 + 5s^2 + 4s + 20 + 20ks = 0$$

- By defining $20k = K$ and dividing by the sum of terms that do not contain $K$,

$$\underbrace{s^3 + 5s^2 + 4s + 20}_{} + Ks = 0$$

we get:

$$1 + K\frac{s}{s^3 + 5s^2 + 4s + 20} = 0$$

- Sketch the root locus of the system given from the new characteristic equation.

# Root locus. Example

- The characteristic equation can be written as:

$$1 + K\frac{s}{(s+5)(s^2+4)} = 0$$

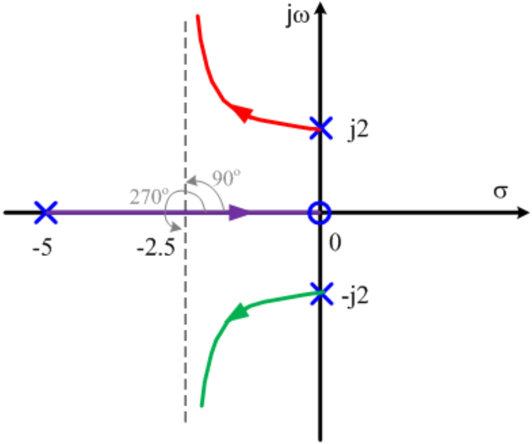
- Open-loop poles: $p_{1,2} = \pm j2$, $p_3 = -5$ and open-loop zeros: $z_1 = 0$. $\Rightarrow n_z = 1$, $n_p = 3$
- The root locus will have $n_p = 3$ branches.
- On the real axis: RL is between the pole $-5$ and the zero at the origin. The other two branches start at the poles $\pm j2$ and approach the asymptotes for increasing $K$.
- The center of the asymptotes:

$$\sigma_A = \frac{-5 - j2 + j2 - 0}{2} = -\frac{5}{2} = -2.5$$

and the angles of the asymptotes:

$$\Phi_A = \frac{\pm 180^o(2q+1)}{2}, \quad q = 0, 1, \quad \Rightarrow \Phi_A = 90^o, 270^o$$

# Root locus. Example



# Root locus. Example. Analysis

- For $k = 0$ the closed-loop system is critically stable because two close-loop poles are on the imaginary axis (the same as the open-loop poles)
- For any $k > 0$ all three closed-loop poles are on the left half s-plane. The system is stable.
- For any $k > 0$ the closed-loop system will have two complex poles with negative real part and one real negative pole. The system is underdamped and the system response is oscillatory