

Frequency Response Analysis

Paula Raica

Department of Automation

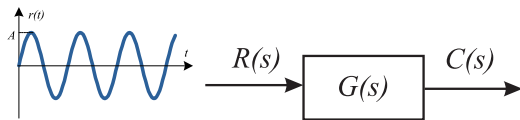
71-73 Dorobantilor Str., room C21, tel: 0264 - 401267

26-28 Baritiu Str., room M14, tel: 0264 - 401239

email: Paula.Raica@aut.utcluj.ro

Technical University of Cluj-Napoca

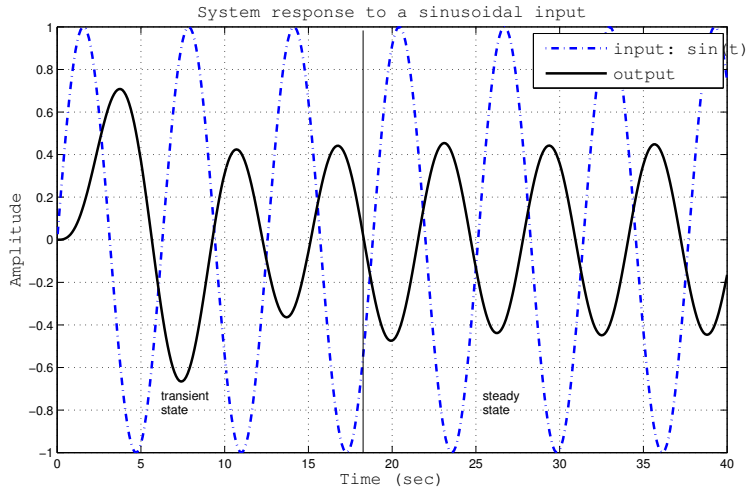
Introduction



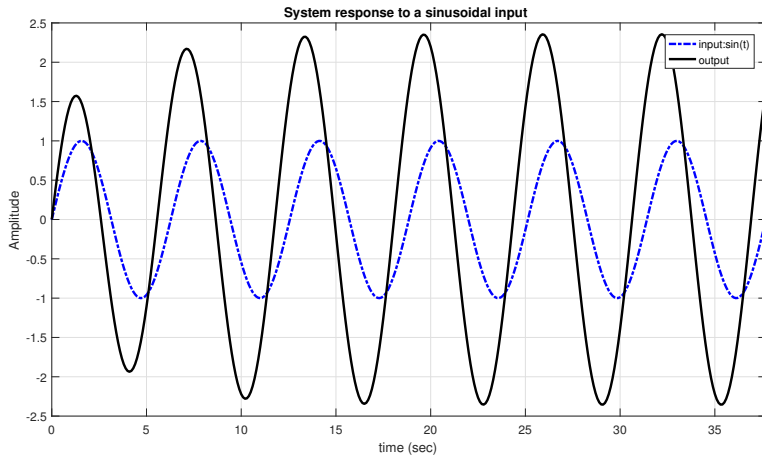
- Analyze the steady-state response of a system with sinusoidal input as the frequency of the sinusoid ω is varied.
- Examine the transfer function $G(j\omega)$ and develop a form of plotting when ω is varied.

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal and the resulting output signal, for a linear system, is sinusoidal in the steady-state, with the same frequency. It differs from the input waveform only in amplitude and phase angle.

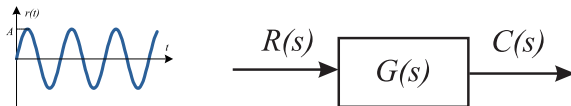
Example



Example



Frequency response



- The input signal and its Laplace transform:

$$r(t) = A \sin \omega t, \quad R(s) = \frac{A\omega}{s^2 + \omega^2}$$

- For a stable system with the transfer function:

$$G(s) = \frac{m(s)}{\prod_{i=1}^n (s + p_i)}$$

- the output signal is:

$$C(s) = G(s)R(s) = \frac{k_1}{s + p_1} + \dots + \frac{k_1}{s + p_n} + \frac{\alpha s + \beta}{s^2 + \omega^2}$$

Frequency response

- System is stable $\Rightarrow$ p_i have negative nonzero real parts and the terms corresponding to the poles in $c(t)$ will be zero at steady-state, or:

$$\lim_{t \rightarrow \infty} \mathcal{L}^{-1} \left[\frac{k_i}{s + p_i} \right] = 0$$

- For $t \rightarrow \infty$ (the steady -state), we can determine that:

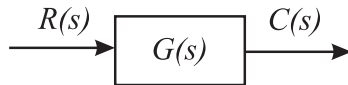
$$c_{ss}(t) = A|G(j\omega)|\sin(\omega t + \varphi)$$

where $\varphi = \angle G(j\omega)$.

See the examples and detailed calculus from the lecture notes Chap6.pdf

Frequency response

- For a system described by the transfer function $G(s)$,
- with a sinusoidal input: $r(t) = A \sin \omega t$
- the output signal **at steady state** is:



$$c_{ss}(t) = A|G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

$$c_{ss}(t) = A \cdot M \sin(\omega t + \varphi)$$

where:

- A is the magnitude of the input signal
- ω is the frequency of the input signal (rad/s)
- $G(j\omega)$ is the sinusoidal transfer function (the transfer function $G(s)$ where $s \rightarrow j\omega$)
- the magnitude: $M = |G(j\omega)|$
- the phase angle: $\varphi = \angle G(j\omega)$

Example

- Consider a first-order system with the transfer function:

$$G(s) = \frac{1}{s + 1}$$

- The input is $r(t) = \sin t$, (the magnitude $A = 1$, the frequency $\omega = 1$ rad/s)
- The output at steady-state:

$$c_{ss}(t) = A \underbrace{|G(j\omega)|}_M \sin(\omega t + \underbrace{\angle G(j\omega)}_\varphi)$$

- Determine:

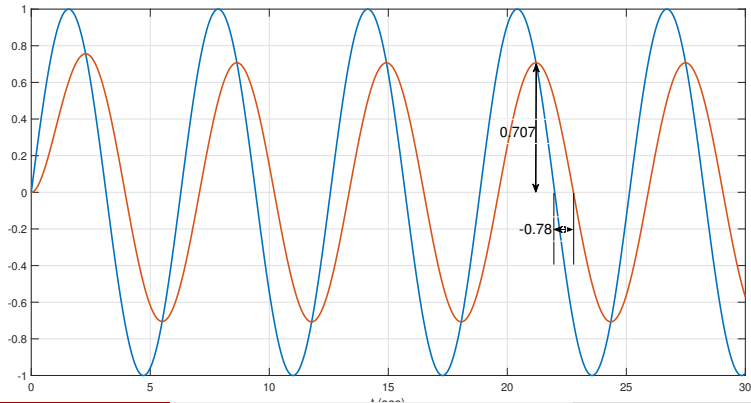
$$G(j \cdot 1) = \frac{1}{j + 1}, \quad \Rightarrow \quad M = |G(j \cdot 1)| = \frac{1}{\sqrt{2}} = 0.707$$

$$\varphi = \angle G(j \cdot 1) = \angle \frac{1}{j + 1} = \angle 1 - \angle j + 1 = -\arctan 1 = -0.78$$

Example

- The output at steady-state is:

$$c_{ss}(t) = \underbrace{1}_A \cdot \underbrace{0.707}_M \sin\left(\underbrace{1}_\omega \cdot t - \underbrace{0.78}_\varphi\right) = 0.707 \sin(t - 0.78)$$



Frequency response

$$c_{ss}(t) = A \underbrace{|G(j\omega)|}_M \sin(\omega t + \underbrace{\angle G(j\omega)}_\varphi)$$

- $M > 1$: the output is amplified with respect to the input
- $M < 1$: the output is attenuated with respect to the input
- $\varphi > 0$: phase lead
- $\varphi < 0$: phase lag

For a sinusoidal transfer function $G(j\omega)$, and $\omega \in (0, \infty)$ represent graphically the frequency response:

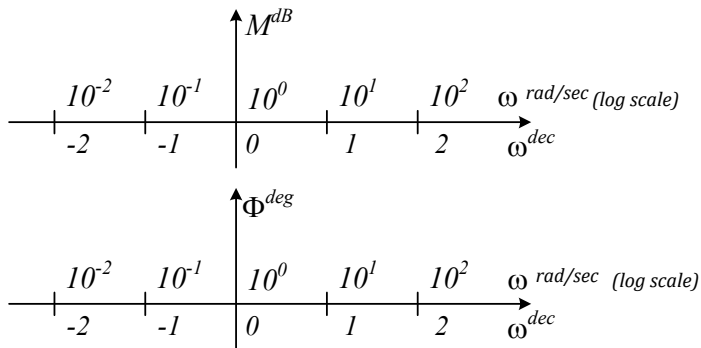
- the magnitude $M = |G(j\omega)|$ versus frequency ω
- the phase angle $\varphi = \angle G(j\omega)$ versus frequency ω

Logarithmic plots / Bode diagrams

Hendrik Wade Bode

■ A Bode (logarithmic) plot consists of:

- a plot of the magnitude $G(j\omega)$ in decibels: $M^{dB} = |G(j\omega)|^{dB} = 20\log_{10}|G(j\omega)|$
- a plot of the phase angle $\angle G(j\omega)$ versus frequency $\omega^{dec} = \log_{10}\omega$ (logarithmic scale). $\phi(\omega)$ in degrees or radians



Bode plots. **dB = decibel**

■ If

$$M = |G(j\omega)| = \frac{A \cdot M}{A} = \frac{\text{magnitude of the output signal}}{\text{magnitude of the input signal}}$$

$$M^{dB} = 20 \log_{10} M$$

■ Example:

$$M = 10 \Rightarrow M^{dB} = 20 \log_{10} 10 = 20 \text{ dB}$$

$$M = 100 \Rightarrow M^{dB} = 20 \log_{10} 10^2 = 40 \text{ dB}$$

$$M = 1 \Rightarrow M^{dB} = 20 \log_{10} 1 = 0 \text{ dB}$$

$$M = 0.1 \Rightarrow M^{dB} = 20 \log_{10} \frac{1}{10} = -20 \text{ dB}$$

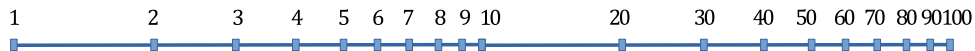
$$M = \frac{\sqrt{2}}{2} \Rightarrow M^{dB} = 20 \log_{10} \frac{\sqrt{2}}{2} = -3.01 \text{ dB}$$

Bode plots. Log-scale

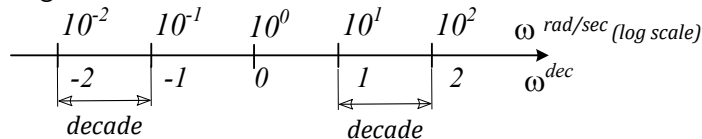
- Between $(10^0, 10^1)$



- Between $(10^0, 10^2)$



- Log-scale. Decade



Bode diagrams. Log-magnitude

The general form of a sinusoidal transfer function is:

$$G(j\omega) = \frac{k \prod_{i=1}^{m_1} (T_i(j\omega) + 1) \prod_{p=1}^{m_2} (\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega) + 1)}{(j\omega)^n \prod_{l=1}^{n_1} (T_l(j\omega) + 1) \prod_{k=1}^{n_2} (\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega) + 1)}$$

$$\begin{aligned} M^{dB} = |G(j\omega)|^{dB} = & 20 \log k + 20 \sum_{i=1}^{m_1} \log |T_i(j\omega) + 1| + \\ & + 20 \sum_{p=1}^{m_2} \log |(\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega) + 1)| - 20 \log |j\omega|^n - \\ & 20 \sum_{l=1}^{n_1} \log |T_l(j\omega) + 1| - 20 \sum_{k=1}^{n_2} \log |(\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega) + 1)| \end{aligned}$$

Bode plots - phase angle

The phase angle of $G(j\omega)$ can be calculated as:

$$\begin{aligned}\Phi = \angle(G(j\omega)) = & \angle \frac{k}{(j\omega)^n} + \sum_{i=1}^{m_1} \angle(T_i(j\omega) + 1) + \\ & + \sum_{p=1}^{m_2} \angle\left(\left(\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega) + 1\right)\right) - \sum_{l=1}^{n_1} \angle(T_l(j\omega) + 1) - \\ & - \sum_{k=1}^{n_2} \angle\left(\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega) + 1\right)\end{aligned}$$

Bode plots

Therefore three different kinds of factors that may occur in a transfer function are as follows:

- 1 Gain k and the integral or derivative factors $\frac{k}{(j\omega)^n}$, (n can be positive or negative)
- 2 First-order factors, $T(j\omega) + 1$, $\frac{1}{T(j\omega) + 1}$
- 3 Quadratic factors $\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1$, or $\frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1}$ with complex-conjugate roots.

It is possible to construct composite plots for any general form of $G(j\omega)$ by sketching the plot for each factor and adding individual curves graphically.

Bode plots

Example:

$$\blacksquare G_1(s) = \frac{100}{s^2(10^2 s^2 + s + 1)}$$

$$\frac{100}{s^2} \text{ of the form } \frac{k}{s^n} \Rightarrow k = 100, n = 2$$

$$\frac{1}{10^2 s^2 + s + 1} = \frac{1}{\frac{1}{\omega_n^2} s^2 + \frac{2\zeta}{\omega_n} s + 1} \Rightarrow \omega_n = 10^{-1}, \zeta = 0.05$$

$$\blacksquare G_2(s) = \frac{s}{10(0.2s + 1)}$$

$$\frac{s}{10} = \frac{10^{-1}}{s^{-1}} \text{ of the form } \frac{k}{s^n} \Rightarrow k = 10^{-1}, n = -1$$

$$\frac{1}{0.2s + 1} \text{ of the form } \frac{1}{Ts + 1} \Rightarrow T = 0.2$$

Diagrame Bode

Example:

$$\blacksquare G_3(s) = \frac{100(s+1)}{2s+1}$$

$$100 \text{ of the form } \frac{k}{s^n} \Rightarrow k = 100, n = 0$$

$$s+1 \text{ of the form } T_1s+1 \Rightarrow T_1 = 1$$

$$\frac{1}{2s+1} \text{ of the form } \frac{1}{T_2s+1} \Rightarrow T_2 = 2$$

$$\blacksquare G_4(s) = \frac{s^2}{9s^2+2s+1}$$

$$s^2 = \frac{1}{s^{-2}} \text{ of the form } \frac{k}{s^n} \Rightarrow k = 1, n = -2$$

$$\frac{1}{9s^2+2s+1} \text{ of the form } \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} \Rightarrow \omega_n = \frac{1}{3}, \zeta = \frac{1}{3}$$

Bode plots

$$G_1(j\omega) = \frac{k}{(j\omega)^n}$$

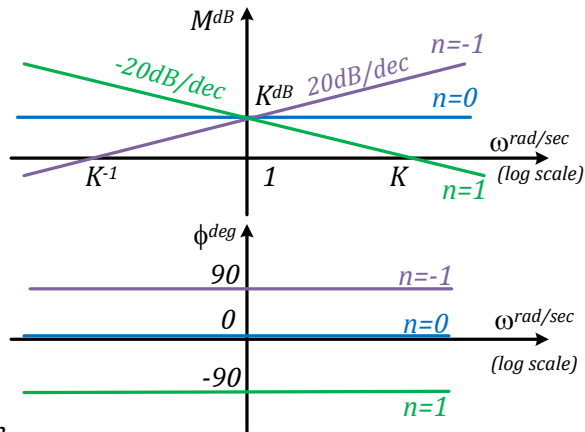
- $M_1^{dB} = 20 \log_{10} \left| \frac{k}{(j\omega)^n} \right|$

$$M_1^{dB} = 20 \log k - 20n \log_{10} \omega$$

= the equation of a straight line, which for $\omega = 1$ is equal to k^{dB} and has the slope $-20n$ (dB/dec).

- The phase angle:

$$\Phi_1 = \angle k - \angle(j\omega^n) = -n \cdot \arctan \frac{\omega}{0} = -90^\circ \cdot n$$



Bode plots

$$G_2(\omega) = \frac{1}{T(j\omega) + 1}$$

$$M_2^{dB} = 20 \log \left| \frac{1}{Tj\omega + 1} \right| = -20 \log \sqrt{T^2 \omega^2 + 1}$$

- For low frequencies $\omega \ll \frac{1}{T}$ (the asymptote at low frequencies)

$$M_2^{dB}|_{\omega \ll} \cong -20 \log 1 = 0 \text{ dB}$$

- For high frequencies $\omega \gg \frac{1}{T}$ (asymptote at high frequencies)

$$M_2^{dB}|_{\omega \gg} \cong -20 \log_{10} \omega T \text{ dB} = -20 \log_{10} \omega - 20 \log_{10} T$$

Equation of a line with the slope -20 (dB/dec)

- The *corner* frequency: $0 = -20 \log_{10} \omega_c - 20 \log_{10} T \Rightarrow \omega_c = \frac{1}{T}$

- The error in the magnitude curve occurs at the corner frequency:

$$M_2^{dB}|_{\omega=\omega_c} = -20\log\sqrt{1 + T^2 \cdot \frac{1}{T^2}} = -20\log\sqrt{2} \cong -3.03 \text{ dB}$$

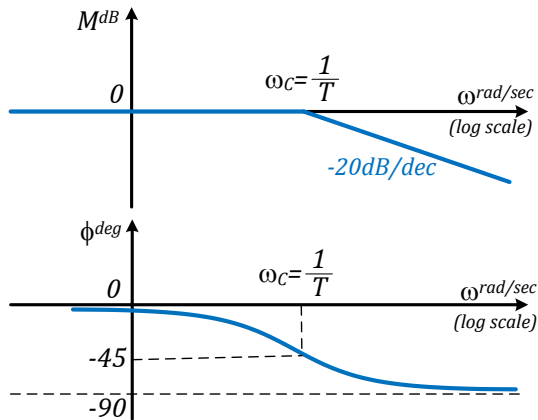
- The phase angle Φ_2 of a factor $\frac{1}{Tj\omega + 1}$ is:

$$\Phi_2 = -\arctan \omega T, \quad \Phi_2|_{\omega=0} = 0^\circ$$

$$\Phi_2|_{\omega=\omega_c} = -\arctan \frac{T}{T} = -45^\circ, \quad \Phi_2|_{\omega=\infty} = -\arctan \infty = -90^\circ$$

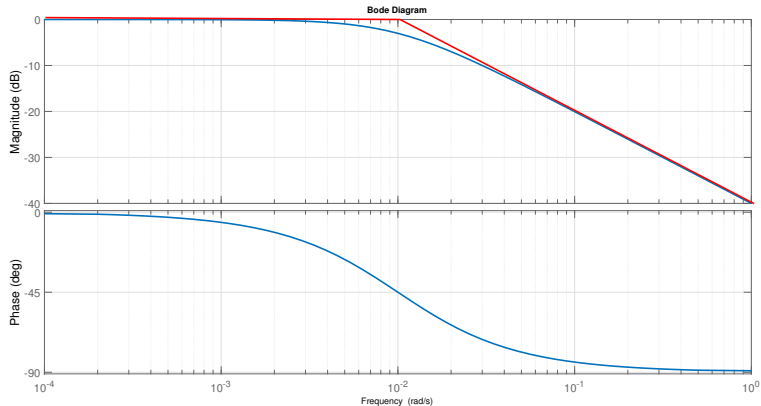
Bode plots

Logarithmic plot for $\frac{1}{Tj\omega + 1}$



Bode plot. Example

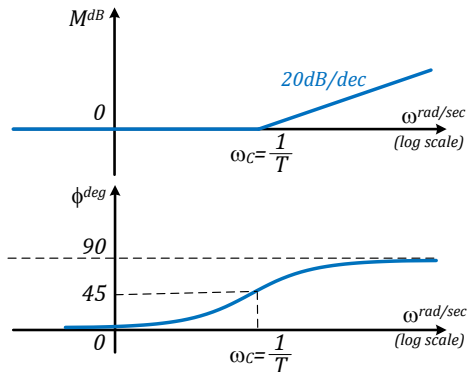
$$G(s) = \frac{1}{100s + 1}, \quad T = 100, \quad \omega_c = \frac{1}{100} = 10^{-2}, \quad \text{slope } -20\text{dB/dec}$$



Bode plots

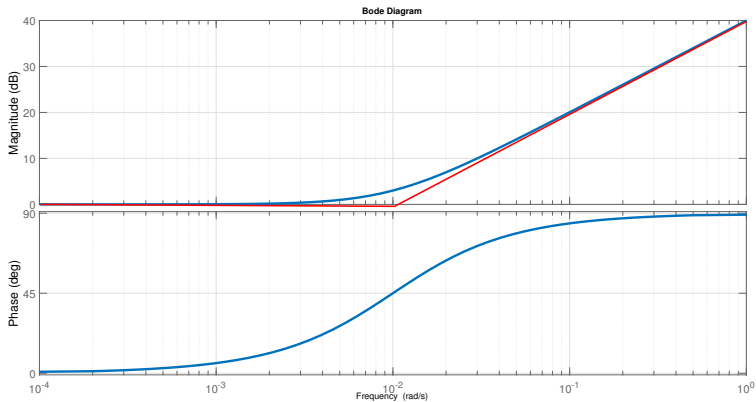
$$20\log |Tj\omega + 1| = -20\log \left| \frac{1}{Tj\omega + 1} \right|, \quad \angle(Tj\omega + 1) = -\angle \left(\frac{1}{Tj\omega + 1} \right)$$

Logarithmic plot for $Tj\omega + 1$



Bode plot. Example

$$G(s) = 100s + 1, \quad T = 100, \quad \omega_c = \frac{1}{100} = 10^{-2}, \quad \text{slope } 20\text{dB/dec}$$



Bode plots

$$G_3(j\omega) = \frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1}$$

$$M_3^{dB} = -20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}$$

- Low frequencies $\omega \ll \omega_n$

$$M_3^{dB}|_{\omega \ll} = -20 \log 1 = 0 \text{ dB}$$

- High frequencies $\omega \gg \omega_n$

$$M_3^{dB}|_{\omega \gg} = -20 \log \frac{\omega^2}{\omega_n^2} = -40 \log \omega - 40 \log \omega_n$$

The equation of a line with the slope -40 dB/dec .

- The corner frequency ω_c : $-40 \log \omega_c - 40 \log \omega_n = 0, \Rightarrow \omega_c = \omega_n$

Bode plots

- Near the corner frequency ω_c , a resonant peak occurs = the maximum error:

$$\begin{aligned}M_3^{dB}|_{\omega=\omega_c} &= -20\log\sqrt{\left(1 - \frac{\omega_n^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega_n}{\omega_n}\right)^2} \\&= -20\log(2\zeta) = -20\log 2 - 20\log\zeta \cong -6^{dB} - \zeta^{dB}\end{aligned}$$

- The phase angle of the quadratic factor

$$\Phi_3 = \angle \left(\frac{1}{\left(\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1\right)} \right) = -\arctan \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$\Phi_3|_{\omega=0} = 0, \quad \Phi_3|_{\omega=\omega_c} = -90^\circ, \quad \Phi_3|_{\omega=\infty} = -180^\circ,$$

Bode plots

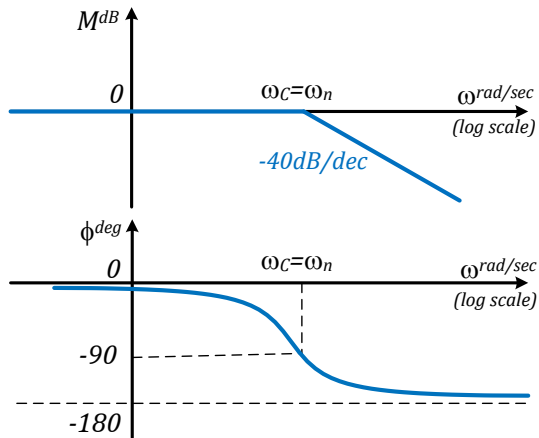
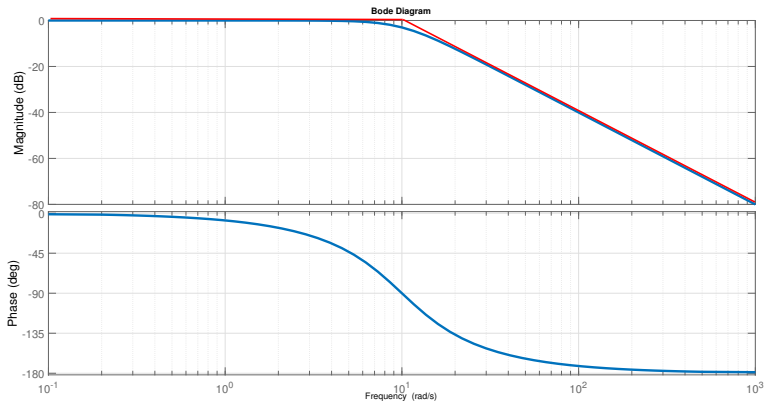


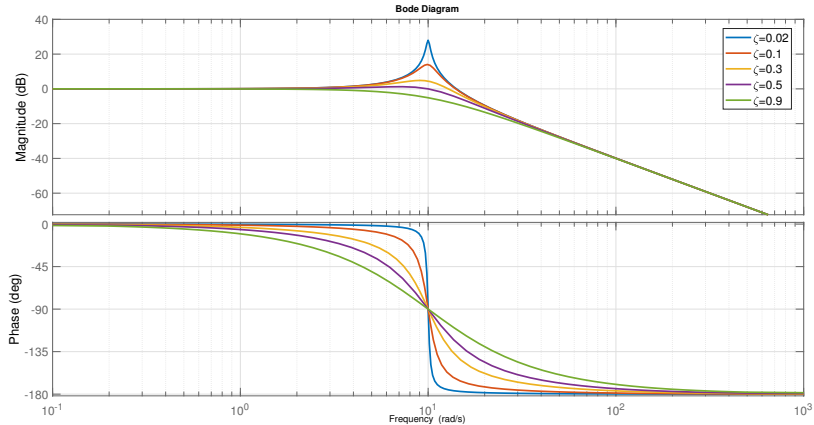
Figure: Logarithmic plot for second-order factors (denominator)

Bode plot. Example

$$G(s) = \frac{1}{10^{-2}s^2 + 0.1414s + 1}, \quad \omega_n = 10, \quad \omega_c = 10, \quad \text{slope } -40\text{dB/dec}$$

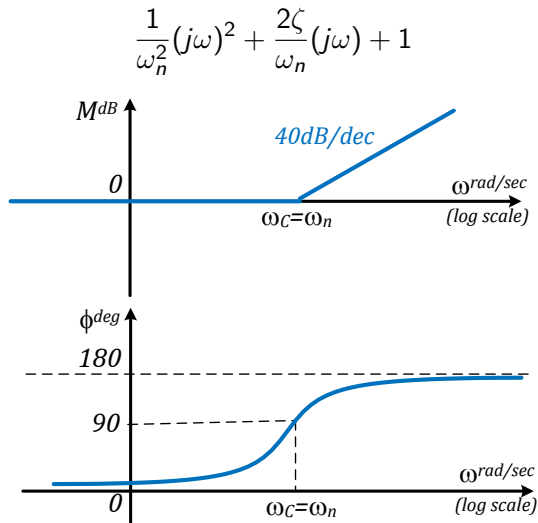


Bode plot. Influence of the damping factor



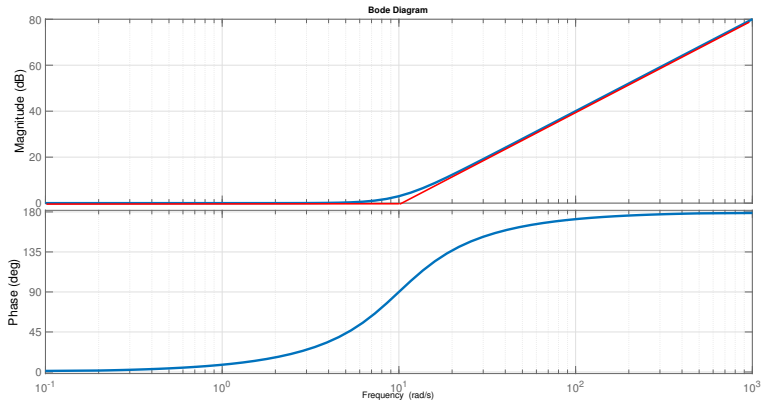
$$M_3^{dB}|_{\omega=\omega_c} = -6^{dB} - \zeta^{dB}$$

Bode plots



Bode plot. Example

$$G(s) = 10^{-2}s^2 + 0.1414s + 1, \omega_n = 10, \omega_c = 10, \text{ slope } 40\text{dB/dec}$$

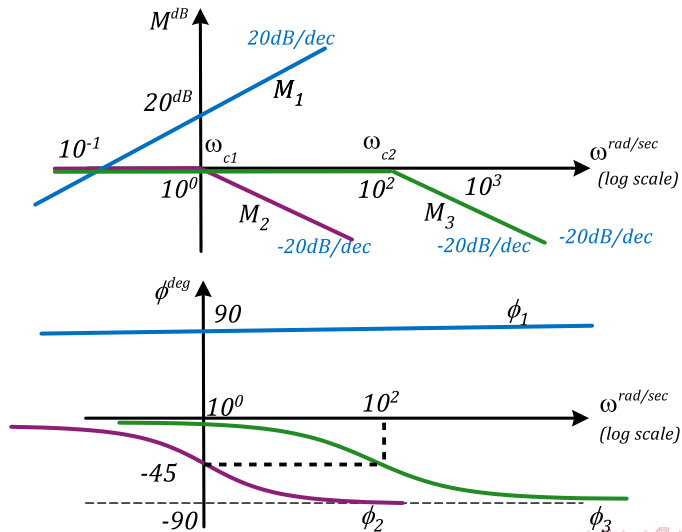


Bode plot. Example

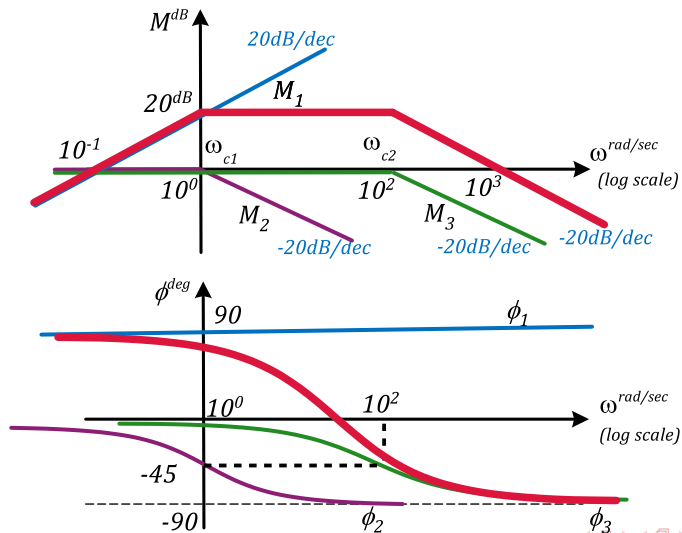
$$G(s) = \frac{10s}{(s+1)(10^{-2}s+1)}$$

- $G_1(s) = 10s = \frac{10}{s^{-1}}$, $k = 10$, $n = -1$, $M_1^{dB}|_{\omega=10^0} = k^{dB} = 20 \log_{10} 10 = 20dB$, slope = 20dB/dec
 $\phi_1 = -90 \cdot n = 90^\circ$
- $G_2(s) = \frac{1}{s+1}$, $T_1 = 1$, $\omega_{c1} = \frac{1}{T_1} = 1$, slope = -20dB/dec
 $\phi_2 \in (0, -90^\circ)$ (arctan), inflection: $(\omega_{c1} = 1, -45^\circ)$
- $G_3(s) = \frac{1}{10^{-2}s+1}$, $T_2 = 10^{-2}$, $\omega_{c2} = 10^2$, slope = -20dB/dec
 $\phi_3 \in (0, -90^\circ)$ (arctan), inflection: $(\omega_{c2} = 10^2, -45^\circ)$

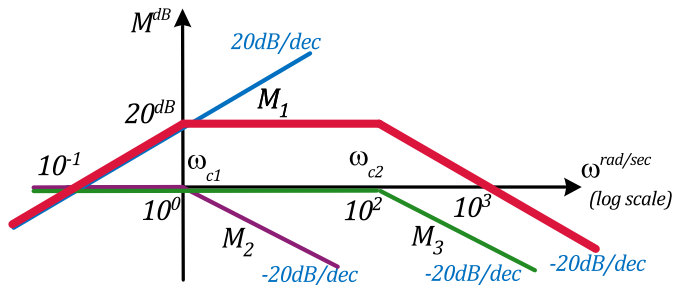
Bode plot. Example



Bode plot. Example. Result

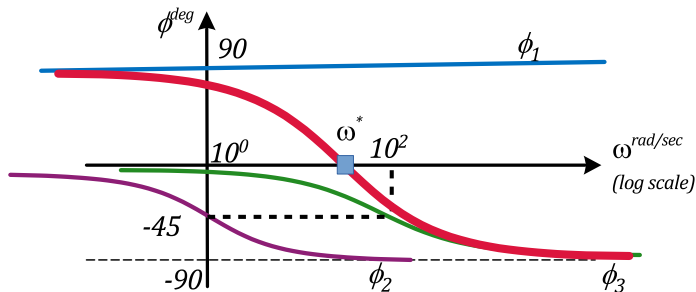


Bode plot. Example. Analysis



- $\omega^{rad/sec} \in (0, 10^{-1}) \cup (10^3, \infty) \Rightarrow M^{dB} < 0, M < 1$, for these frequencies of the input, the system attenuates the sinusoidal inputs
- $\omega^{rad/sec} \in (10^{-1}, 10^3) \Rightarrow M^{dB} > 0, M > 1$ for these frequencies of the input the system amplifies the sinusoidal inputs
- $\omega^{rad/sec} = 10^{-1}, \omega^{rad/sec} = 10^3, M^{dB} = 0, M = 1$ magnitude of the output = magnitude of the input

Bode plot. Example. Analysis



■ $\omega^{rad/sec} \in (0, \omega^*) \Rightarrow \phi > 0$ phase lead

■ $\omega^{rad/sec} \in (\omega^*, \infty) \Rightarrow \phi < 0$ phase lag

Bode diagrams. Example

Draw the Bode diagram for the following transfer function:

$$G(s) = \frac{10^3(s + 10)}{s(s + 1)(s^2 + 10s + 100)} = \frac{10^2(\frac{1}{10}s + 1)}{s(s + 1)(\frac{1}{100}s^2 + \frac{1}{10}s + 1)}$$

$$G_1(j\omega) = \frac{10^2}{j\omega} = \frac{k}{(j\omega)^1};$$

$$G_2(j\omega) = \frac{1}{10}(j\omega) + 1 = T_1(j\omega) + 1;$$

$$G_3(j\omega) = \frac{1}{j\omega + 1} = \frac{1}{T_2j\omega + 1};$$

$$G_4(j\omega) = \frac{1}{\frac{1}{100}(j\omega)^2 + \frac{1}{10}(j\omega) + 1} = \frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1}$$

Bode diagrams. Example

The Bode diagram is plotted for every one of these factors and than the curves are added graphically.

$$k = 10^2; \quad T_1 = \frac{1}{10}; \quad T_2 = 1; \quad \omega_n = 10; \quad \zeta = 0.5.$$

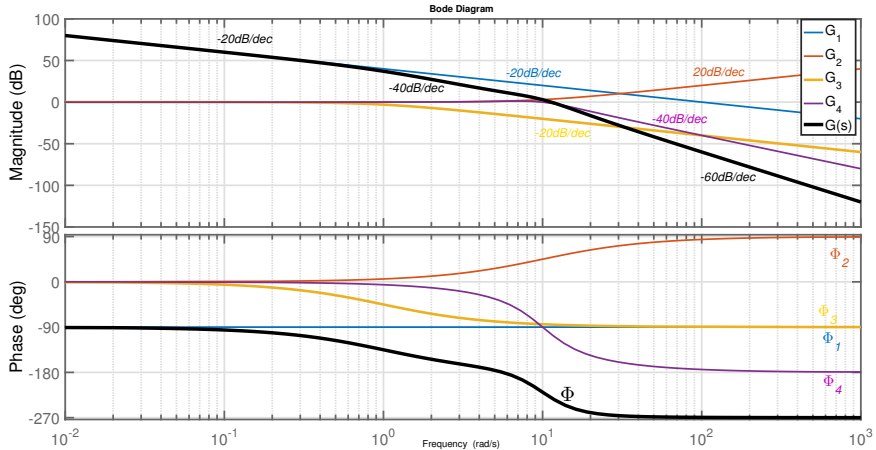
$$k^{dB} = 20 \log 10^2 = 40 \text{ dB}, \quad \Phi_1 = -90^\circ$$

$$\omega_{c1}^{\text{rad/sec}} = \frac{1}{T_1} = \frac{1}{1/10} = 10 \text{ rad/sec}, \quad \Phi_2 \in [0, 90^\circ]$$

$$\omega_{c2}^{\text{rad/sec}} = \frac{1}{T_2} = 1 \text{ rad/sec}, \quad \Phi_3 \in [0, -90^\circ]$$

$$\omega_{c3}^{\text{rad/sec}} = \omega_n = 10 \text{ rad/sec}, \quad \Phi_4 \in [0, -180^\circ]$$

Bode diagrams. Example



Reading Bode diagrams

- If $|G(j\omega)| > 1$ (or $M^{dB} = |G(j\omega)|^{dB} > 0$), the output will be amplified.
- If $|G(j\omega)| < 1$ (or $M^{dB} = |G(j\omega)|^{dB} < 0$), the output is attenuated.
- If $\Phi > 0$ the output is shifted with a positive angle comparing to the input and the system has phase lead.
- If $\Phi < 0$ the output is shifted with a negative angle with respect to the input (phase lag).

See BodeRules.pdf