

Frequency response analysis. Applications

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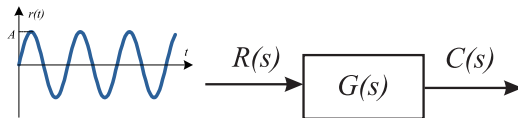
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Introduction



input signal: $r(t) = A \sin \omega t$

Frequency response:

$$c_{ss}(t) = A \underbrace{|G(j\omega)|}_M \sin(\omega t + \underbrace{\angle G(j\omega)}_\varphi)$$

- the magnitude $M = |G(j\omega)|$
- the phase angle $\varphi = \angle G(j\omega)$

Frequency response specifications

Consider the standard second order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\frac{1}{\omega_n}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

The sinusoidal transfer function:

$$G(j\omega) = \frac{1}{\frac{1}{\omega_n}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + 2j\zeta\frac{\omega}{\omega_n}}$$

with the magnitude and phase angle:

$$M = |G(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}, \quad \varphi = \angle G(j\omega) = \operatorname{atan}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

Frequency response specifications - resonant frequency

- If $M = |G(j\omega)|$ has a peak value at some frequency, this is called *resonant frequency*.
- Max $M = \max |G(j\omega)|$ will occur when the denominator is minimum. The resonant frequency ω_r is obtained from $dM/d\omega = 0$:

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

for $0 \leq \zeta \leq \frac{1}{\sqrt{2}} = 0.707$ (because the quantity under the square root must be positive for real values of ω)

- Notice that for $\zeta \rightarrow 0$, $\omega_r \rightarrow \omega_n$
- For $\zeta > 0.707$ there is no resonant peak. The magnitude of $G(j\omega)$ decreases monotonically with ω .

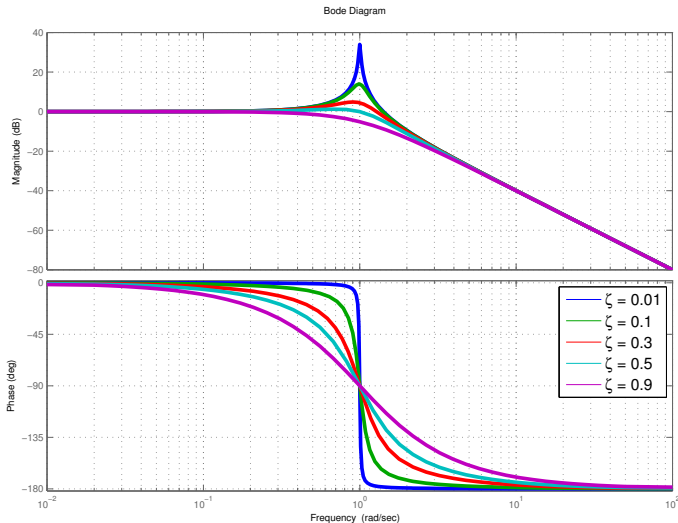
Resonant peak

- The *resonant peak* M_r occurs at the *peak* (or *resonant*) frequency ω_r .
- The magnitude of *the resonant peak* is obtained by substituting ω_r into M .
- For $0 \leq \zeta \leq 0.707$:

$$M_r = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

- For $0.707 \leq \zeta \leq 1$: $M_r = 1$
- If $\zeta \rightarrow 0$, $M_r \rightarrow \infty$.
- The magnitude of the resonant peak - indication of the relative stability of the system. A large M_r indicates the presence of a pair of dominant poles with small damping ratio (undesirable transient response).

Example. Various values of the damping factor



$$G_1(s) = \frac{1}{s^2 + 0.02s + 1}$$

$$G_2(s) = \frac{1}{s^2 + 0.2s + 1}$$

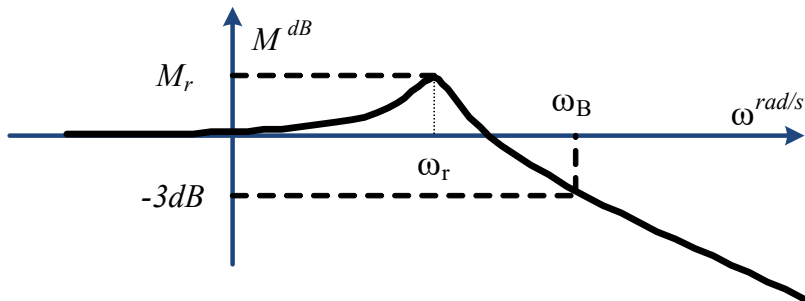
$$G_3(s) = \frac{1}{s^2 + 0.6s + 1}$$

$$G_4(s) = \frac{1}{s^2 + s + 1}$$

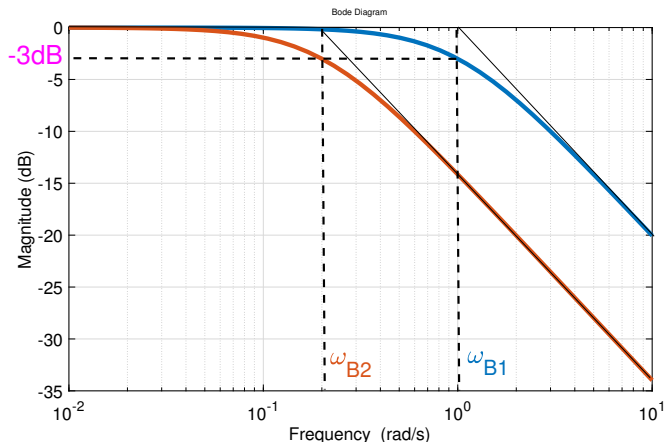
$$G_5(s) = \frac{1}{s^2 + 1.8s + 1}$$

The bandwidth

- The bandwidth = frequency at which the frequency response has declined -3 dB from 0dB.
- The *bandwidth* ω_B is a measure of the system's ability to faithfully reproduce an input signal.



Bandwidth. Example



System 1 has a larger bandwidth than System 2.

■ Blue

$$G_1(s) = \frac{1}{s + 1}$$

$$T_1 = 1$$

$$\omega_{B1} = 1 \text{ rad/s}$$

■ Red

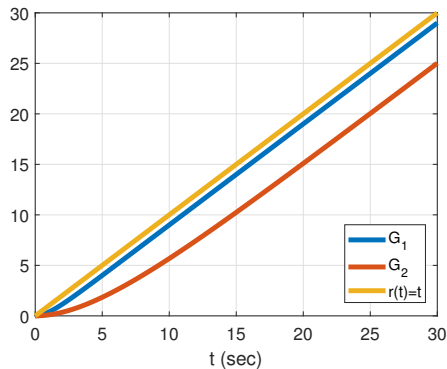
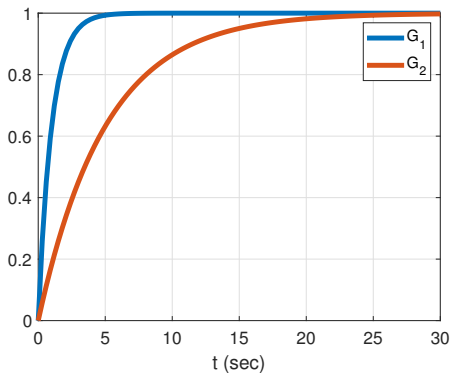
$$G_2(s) = \frac{1}{5s + 1}$$

$$T_2 = 5$$

$$\omega_{B2} = 1/5 = 0.2 \text{ rad/s}$$

Bandwidth. Example

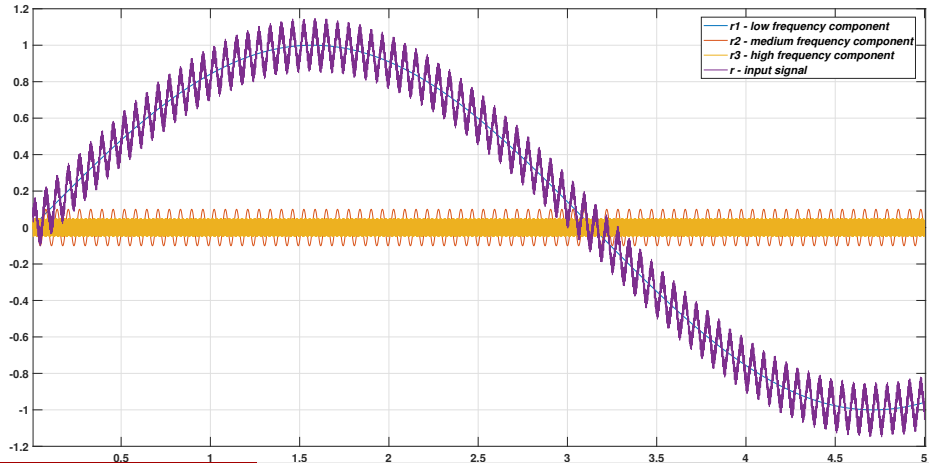
The system with the larger bandwidth provides the faster step response and the higher fidelity ramp response.



Application to filters

The input has 3 frequency components

$$r(t) = r_1 + r_2 + r_3 = \sin(t) + 0.1\sin(100t) + 0.05\sin(10000t).$$



Application to filters

The task is to design filters for this type of input, such that:

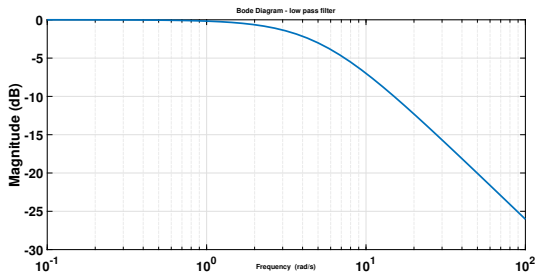
- the high frequency component is rejected
- the low frequency component is rejected
- the medium frequency component is rejected
- the low and high frequency components are rejected

Application to filters - low pass

The low-pass filter from the figure is described by the transfer function

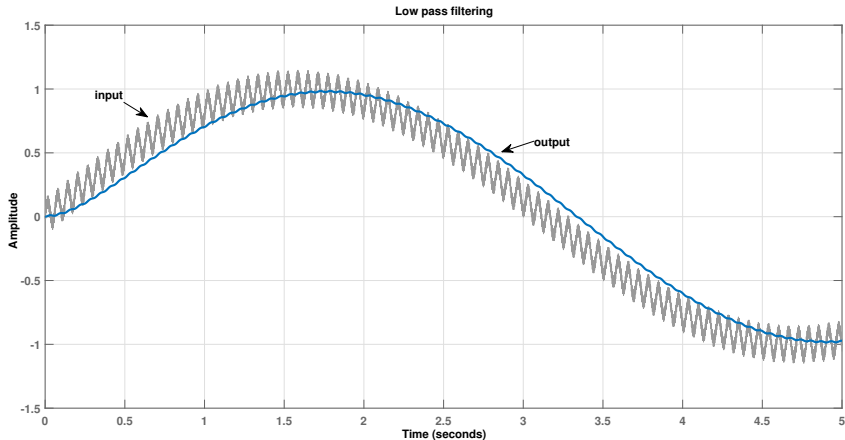
$$G_{LP}(s) = \frac{1}{Ts + 1}$$

The corner frequency is
 $\omega_1 = 5 \text{ rad/sec}$.



Application to filters - low pass

Applying this low-pass filter to our input signal, we get the output from the figure.

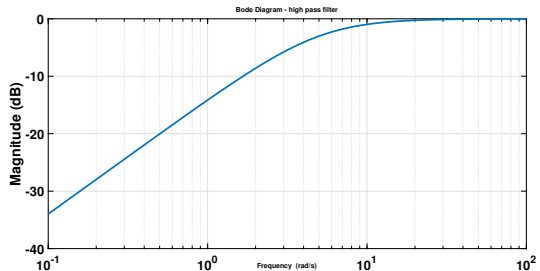


Application to filters - high pass

The high-pass filter from the figure is described by the transfer function

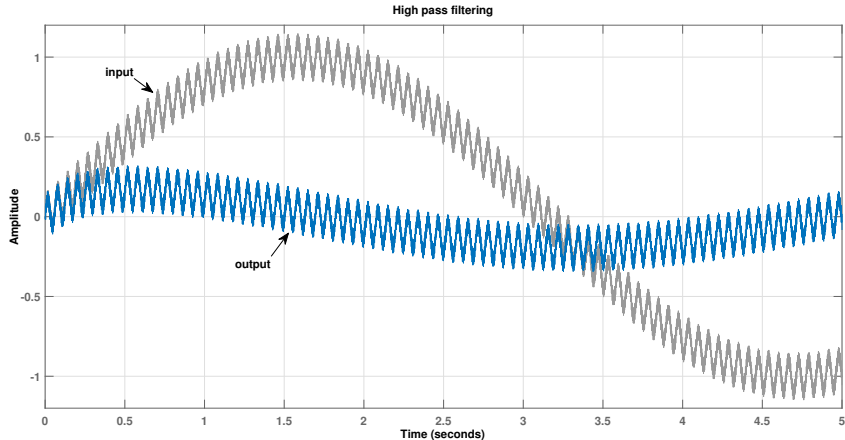
$$G_{HP}(s) = \frac{Ks}{Ts + 1}$$

The corner frequency is
 $\omega_1 = 5 \text{ rad/sec}$.



Application to filters - high pass

Applying this high-pass filter to our input signal, we get the output from the figure.

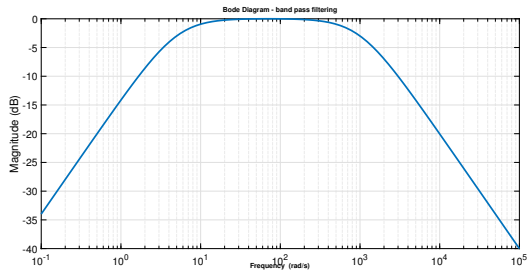


Application to filters - band pass

The band-pass filter from the figure is described by the transfer function

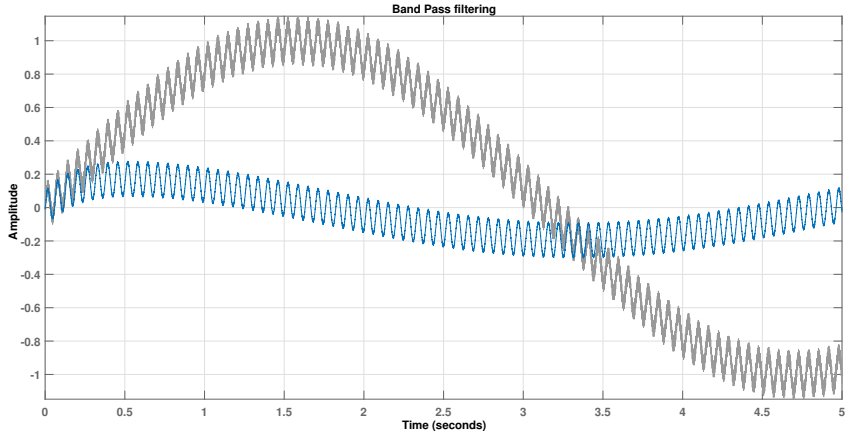
$$G_{BP}(s) = \frac{Ks}{\omega_n^2 s^2 + \frac{2\zeta}{\omega_n} s + 1}$$

The natural frequency is $\omega_n = 100 \text{ rad/sec}$.



Application to analog filters - band pass

Applying this band-pass filter to our input signal, we get the output from the figure.



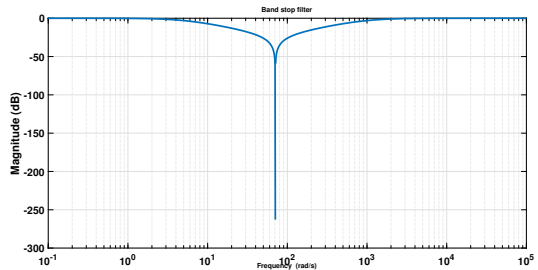
Application to filters - band stop

The band-stop filter from the figure is described by the transfer function

$$G_{BS}(s) = \frac{\frac{1}{\omega_n^2}s^2 + 1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

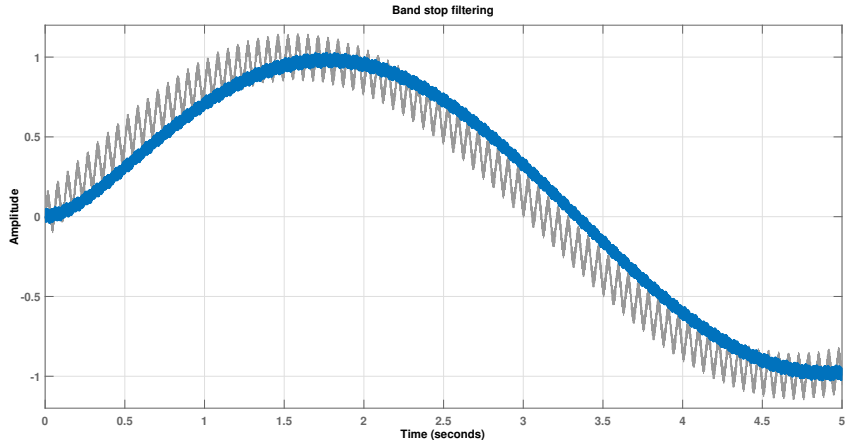
The natural frequency is

$$\omega_n = 100 \text{ rad/sec.}$$



Application to filters - band stop

Applying this band-stop filter to our input signal, we get the output from the figure.



Application to filters - summary

