

Chapter 6

Frequency response

6.1 Introduction

We have examined the use of test signals such as a step, an impulse and a ramp signal. In this chapter we will use a sinusoidal input signal and consider the steady-state response of the system as the frequency of the sinusoid is varied. Thus, we will look at the response of the system to a changing frequency, ω .

We will examine the transfer function $G(j\omega)$, when $s = j\omega$ and develop a form of plotting the complex number for $G(j\omega)$ when ω is varied, (Dorf and Bishop, 2008)

Industrial control systems are often designed by use of frequency-response methods. Frequency-response tests are, in general, simple and can be made accurately by use of readily available sinusoidal generators and precise measurement equipment. Often, the transfer function of complicated components can be determined experimentally by frequency-response tests.

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal and the resulting output signal, for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs from the input waveform only in amplitude and phase angle, (Dorf and Bishop, 2008).

Consider the system shown in Figure 6.1 where the input $r(t)$ is a sinusoid: $r(t) = A\sin\omega t$. The Laplace

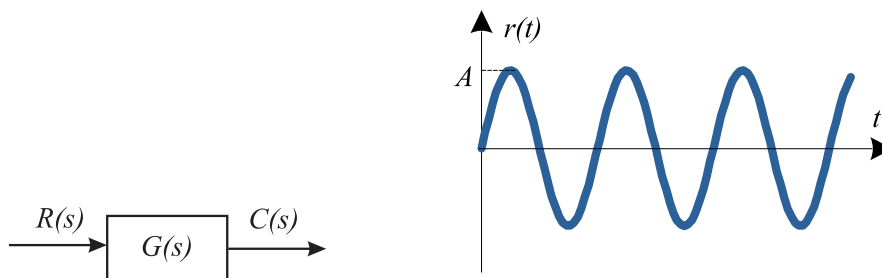


Figure 6.1: System with sinusoidal input

transform of the input is:

$$R(s) = \frac{A\omega}{s^2 + \omega^2}$$

and the transfer function:

$$G(s) = \frac{m(s)}{q(s)} = \frac{m(s)}{\prod_{i=1}^n (s + p_i)}$$

Then, the output $C(s)$ is given by:

$$C(s) = G(s)R(s)$$

or, in partial fraction form:

$$C(s) = \frac{k_1}{s + p_1} + \dots + \frac{k_n}{s + p_n} + \frac{\alpha s + \beta}{s^2 + \omega^2}$$

If the system is stable, then all p_i have negative nonzero real parts and the terms corresponding to the poles in $c(t)$ will be zero at steady-state, or:

$$\lim_{t \rightarrow \infty} \mathcal{L}^{-1} \left[\frac{k_i}{s + p_i} \right] = 0$$

In the limit, for $c(t)$, we obtain for $t \rightarrow \infty$ (the steady -state):

$$c(t) = \mathcal{L}^{-1} \left[\frac{\alpha s + \beta}{s^2 + \omega^2} \right] = \frac{1}{\omega} |A\omega G(j\omega)| \sin(\omega t + \varphi) = A |G(j\omega)| \sin(\omega t + \varphi)$$

where $\varphi = \angle G(j\omega)$.

Example 6.1.1 Consider a first-order system with the transfer function

$$G(s) = \frac{1}{s + 1}$$

The input is sinusoidal: $r(t) = A \sin \omega t$. We shall calculate the output signal $c(t)$ at steady state ($t \rightarrow \infty$).

$$R(s) = \mathcal{L}[A \sin \omega t] = \frac{A\omega}{s^2 + \omega^2}$$

$$C(s) = R(s)G(s) = \frac{A\omega}{(s^2 + \omega^2)(s + 1)} = \frac{A\omega}{\omega^2 + 1} \frac{1}{s + 1} + \frac{A\omega}{\omega^2 + 1} \frac{1 - s}{s^2 + \omega^2}$$

$$c(t) = \mathcal{L}^{-1}[C(s)] = \frac{A\omega}{\omega^2 + 1} \mathcal{L}^{-1} \left[\frac{1}{s + 1} + \frac{1 - s}{s^2 + \omega^2} \right]$$

$$c(t) = \frac{A\omega}{\omega^2 + 1} e^{-t} + \frac{A\omega}{\omega^2 + 1} \mathcal{L}^{-1} \left[\frac{1 - s}{s^2 + \omega^2} \right]$$

At steady state ($t \rightarrow \infty$), the exponential $e^{-t} \rightarrow 0$. The steady state of the output signal is given by:

$$c(t) = \frac{A\omega}{\omega^2 + 1} \mathcal{L}^{-1} \left[\frac{1 - s}{s^2 + \omega^2} \right] = \frac{A\omega}{\omega^2 + 1} \mathcal{L}^{-1} \left[\frac{\omega}{s^2 + \omega^2} \frac{1}{\omega} - \frac{s}{s^2 + \omega^2} \right]$$

$$c(t) = \frac{A\omega}{\omega^2 + 1} \left[\frac{1}{\omega} \sin \omega t - \cos \omega t \right]$$

If we replace $s = j\omega$ into $G(s)$ we obtain a complex quantity with the magnitude and phase angle:

$$|G(j\omega)| = \left| \frac{1}{j\omega + 1} \right| = \frac{1}{\sqrt{\omega^2 + 1}}$$

$$\angle G(j\omega) = \varphi = \angle \frac{1}{j\omega + 1} = -\arctan \omega$$

$$\tan \varphi = -\omega; \quad \frac{\sin \varphi}{\cos \varphi} = -\omega; \quad \cos \varphi = \frac{1}{\sqrt{\omega^2 + 1}}$$

By replacing these results into $c(t)$:

$$c(t) = \frac{A\omega}{\omega^2 + 1} \frac{\sin \omega t - \omega \cos \omega t}{\omega} = \frac{A}{\omega^2 + 1} \frac{1}{\cos \varphi} [\sin \omega t \cos \varphi + \sin \varphi \cos \omega t]$$

$$c(t) = \frac{A}{\omega^2 + 1} \frac{1}{\cos \varphi} \sin(\omega t + \varphi) = \frac{A}{\sqrt{\omega^2 + 1}} \sin(\omega t + \varphi)$$

$$c(t) = A|G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

Thus, the steady-state output signal depends only on the magnitude and phase of $G(j\omega)$ at a specific frequency ω . Notice that the steady-state response as described before is true only for stable systems, $G(s)$.

We see that a stable linear time-invariant system subjected to a sinusoidal input will, at steady-state, have a sinusoidal output of the same frequency as the input. But the amplitude and phase of the output will be different from those of the input. In fact, the amplitude of the output is given by the product of that of the input and $|G(j\omega)|$ while the phase angle differs from that of the input by the amount $\varphi = \angle G(j\omega)$. An example of input and output sinusoidal signals is shown in Figure 6.2.

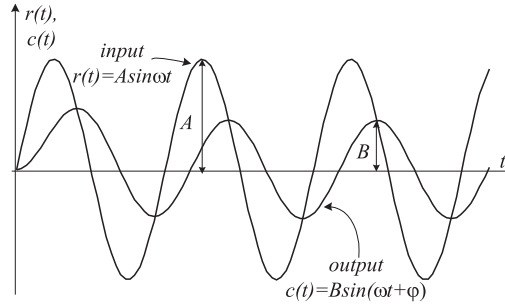


Figure 6.2: Input and output sinusoidal signals

Thus, for sinusoidal inputs the amplitude ratio of the output sinusoid to the input sinusoid is:

$$|G(j\omega)| = \left| \frac{C(j\omega)}{R(j\omega)} \right|$$

and the phase shift of the output sinusoid with respect to the input sinusoid:

$$\angle G(j\omega) = \angle \frac{C(j\omega)}{R(j\omega)}$$

Hence, the response characteristics of a system to a sinusoidal input can be obtained directly from:

$$\frac{C(j\omega)}{R(j\omega)} = G(j\omega)$$

The function $G(j\omega)$ is called **the sinusoidal transfer function**. It is a complex quantity and can be represented by the magnitude and phase angle with frequency as a parameter. A negative phase angle is called **phase lag** and a positive phase angle is called **phase lead**.

The sinusoidal transfer function of a linear system is obtained by substituting $j\omega$ for s in the transfer function of the system. To completely characterize a linear system in the frequency domain, we must specify both the amplitude ratio and the phase angle as functions of the frequency ω .

Example 6.1.2 Consider the system shown in Figure 6.3.

The transfer function $G(s)$ is

$$G(s) = \frac{k}{Ts + 1}$$

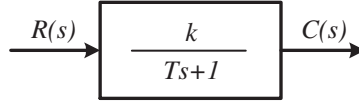


Figure 6.3: First-order system

For the sinusoidal input $r(t) = A \sin \omega t$, the steady-state output $c_{ss}(t)$ can be found as follows:

- Substituting $j\omega$ for s in $G(s)$ yields:

$$G(j\omega) = \frac{k}{jT\omega + 1}$$

- The amplitude ratio of the output is:

$$|G(j\omega)| = \frac{k}{\sqrt{1 + T^2\omega^2}}$$

while the phase angle of the output φ is

$$\varphi = \angle G(j\omega) = -\arctan T\omega$$

Thus, for the input $r(t)$, the steady-state output $c_{ss}(t)$ can be obtained as:

$$c_{ss}(t) = \frac{Ak}{\sqrt{1 + T^2\omega^2}} \sin(\omega t - \arctan T\omega)$$

6.2 Logarithmic plots. Bode diagrams

The sinusoidal transfer function, a complex function of the frequency ω is characterized by its magnitude and phase angle, with frequency as parameter. A sinusoidal transfer function may be represented by two separate plots, one giving the magnitude versus frequency and the other the phase angle versus frequency.

The logarithmic plots are called Bode plots in the honor of H.W.Bode who used them extensively in his studies of feedback amplifiers.

The transfer function in the frequency domain is:

$$G(j\omega) = |G(\omega)|e^{j\Phi(\omega)}$$

A Bode diagram consists of two graphs: one is the plot of the logarithm of the magnitude of a sinusoidal transfer function, and the other one is a plot of the phase angle, both are plotted against the frequency in logarithmic scale, or *decades*:

$$\omega^{dec} = \log_{10}\omega$$

The standard representation of the logarithmic magnitude of $G(j\omega)$, denoted M^{dB} , is $20\log_{10}|G(j\omega)|$. The unit used in this representation is the decibel, usually abbreviated dB :

$$M^{dB} = |G(j\omega)|^{dB} = 20\log_{10}|G(j\omega)|$$

The phase angle of the transfer function, $\phi(\omega)$, is represented (in degrees or radians) versus the logarithmic frequency.

The axes for a Bode diagram are shown in Figure 6.4.

The main advantage of using the logarithmic plot is that multiplication of magnitudes can be converted into addition. Furthermore, a simple method for sketching an approximate log-magnitude curve is available. It is based on asymptotic approximations.

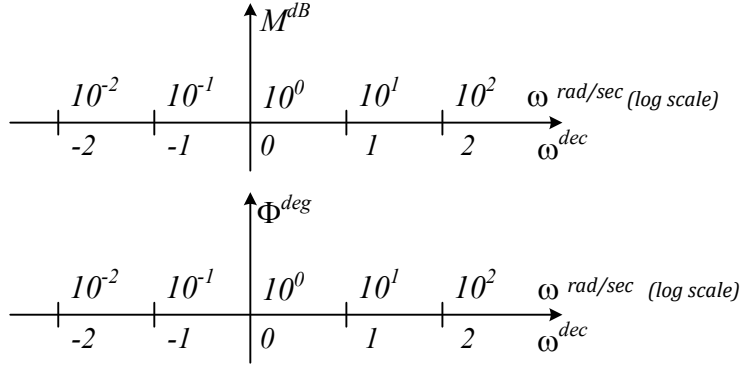


Figure 6.4: Axes for a Bode diagram

The general form of a sinusoidal transfer function emphasizing first and second order factors as well as the poles (or zeros) at the origin, is:

$$G(j\omega) = \frac{k \prod_{i=1}^{m_1} (T_i(j\omega) + 1) \prod_{p=1}^{m_2} (\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega) + 1)}{(j\omega)^n \prod_{l=1}^{n_1} (T_l(j\omega) + 1) \prod_{k=1}^{n_2} (\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega) + 1)}$$

The logarithmic magnitude of $G(j\omega)$, expressed in decibels, is:

$$\begin{aligned} M^{dB} = |G(j\omega)|^{dB} = & 20 \log_{10} k + 20 \sum_{i=1}^{m_1} \log_{10} |T_i(j\omega) + 1| + \\ & + 20 \sum_{p=1}^{m_2} \log_{10} |(\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega) + 1)| - 20 \log_{10} |j\omega|^n - \\ & - 20 \sum_{l=1}^{n_1} \log_{10} |T_l(j\omega) + 1| - 20 \sum_{k=1}^{n_2} \log_{10} |(\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega) + 1)| \end{aligned}$$

The phase angle of $G(j\omega)$ can be calculated as:

$$\begin{aligned} \Phi = \angle(G(j\omega)) = & \angle \frac{k}{(j\omega)^n} + \sum_{i=1}^{m_1} \angle(T_i(j\omega) + 1) + \\ & + \sum_{p=1}^{m_2} \angle((\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega) + 1)) - \sum_{l=1}^{n_1} \angle(T_l(j\omega) + 1) - \\ & - \sum_{k=1}^{n_2} \angle((\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega) + 1)) \end{aligned}$$

Therefore three different kind of factors that may occur in a transfer function are as follows:

1. Gain k and the integral or derivative factors $\frac{k}{(j\omega)^n}$, (n is integer, positive or negative)
2. First-order factors, $T(j\omega) + 1$ or $\frac{1}{T(j\omega) + 1}$
3. Quadratic factors $\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1$ or $\frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1}$

It is possible to construct composite plots for any general form of $G(j\omega)$ by sketching the plot for each factor and adding individual curves graphically because adding the logarithm of the gains correspond to multiplying them together.

1. Gain k and the integral or derivative factors $\frac{k}{(j\omega)^n}$. The logarithmic gain, in decibels, is written as:

$$M_1^{dB} = 20 \log_{10} \left| \frac{k}{(j\omega)^n} \right| = 20 \log_{10} k - 20n \log_{10} \omega = k^{dB} - 20n \cdot \log_{10} \omega$$

In a coordinate system $(M_1^{dB}, \log_{10} \omega)$ as shown in figure 6.4, this is the equation of a straight line which has the slope $-20n$ (dB/dec). For $\omega = 1 = 10^0$, the magnitude in dB is obtained as:

$$M_1^{dB}|_{\omega=10^0} = k^{dB} - 20n \cdot \log_{10} 10^0 = k^{dB}$$

The intersection with the ω axis ($M_1^{dB} = 0$) is obtained from:

$$0 = 20 \log_{10} k - 20n \log_{10} \omega \Rightarrow \log_{10} \omega = \frac{1}{n} \log_{10} k \Rightarrow \omega = k^{1/n}$$

The phase angle is:

$$\Phi_1 = -n \cdot \arctan \frac{\omega}{0} = -90^\circ \cdot n$$

The Bode diagram corresponding to integral or derivative factors plus the gain is shown in Figure 6.5.

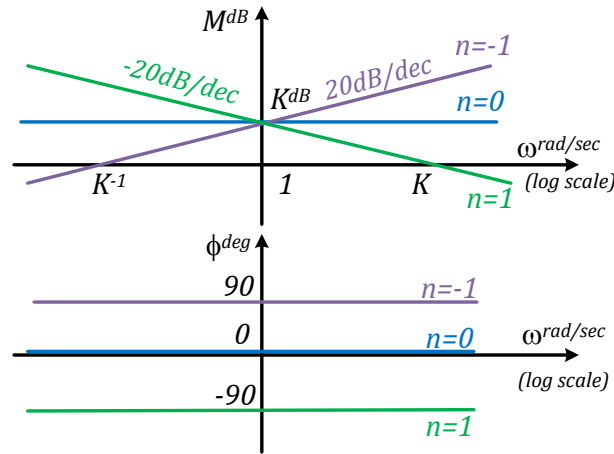


Figure 6.5: Logarithmic plot for integral and derivative factors

2. First-order factors, $T(j\omega) + 1$

The log-magnitude of the first-order factor $T(j\omega) + 1$ is:

$$M_2^{dB} = 20 \log_{10} |Tj\omega + 1| = 20 \log_{10} \sqrt{T^2\omega^2 + 1}$$

For low frequencies, such that $\omega \ll 1/T$, the log-magnitude may be approximated by:

$$M_2^{dB}|_{\omega \ll} \cong 20 \log_{10} 1 = 0 \text{ dB}$$

Thus, the log-magnitude curve at low frequencies is the constant 0 dB line. For high frequencies, such that $\omega \gg 1/T$, the magnitude in decibels may be approximated by:

$$M_2^{dB}|_{\omega \gg} \cong 20 \log_{10} \omega T \text{ dB.}$$

or

$$M_2^{dB}|_{\omega \gg} \cong 20 \cdot \log_{10} \omega + 20 \log_{10} T$$

Since the representation is made for M^{dB} versus $\log_{10} \omega$, this is the equation of a straight line with the slope $20dB/dec$

The frequency ω_c , at which the two asymptotes meet is called the *corner frequency* or *break frequency*, and is calculated from:

$$M_2^{dB}|_{\omega \ll} = M_2^{dB}|_{\omega \gg}$$

or

$$0 = 20 \log_{10} \omega_c + 20 \log_{10} T \Rightarrow \log_{10} \omega_c = -\log_{10} T = \log_{10} \frac{1}{T}$$

and we get:

$$\omega_c = \frac{1}{T}$$

The error in the magnitude curve caused by the use of asymptotes can be calculated. The maximum error occurs at the corner frequency and is calculated by replacing the expression of the corner frequency in the exact relation for log-magnitude:

$$M_2^{dB}|_{\omega=\omega_c} = 20 \log \sqrt{1 + T \cdot \frac{1}{T}} = 20 \log \sqrt{2} \cong 3.03 \text{ dB}$$

Since the asymptotes are quite easy to draw and are sufficiently close to the exact curve, the use of such approximations in drawing Bode diagrams is convenient in establishing the general nature of the frequency-response characteristics.

The exact phase angle Φ_2 of the factor $Tj\omega + 1$ is:

$$\Phi_2 = \arctan \omega T$$

At zero frequency, the phase angle is 0° . At the corner frequency, the phase angle is

$$\Phi_2 = \arctan \frac{T}{T} = \arctan 1 = 45^\circ$$

At infinity, when $\omega \rightarrow \infty$ the phase angle becomes:

$$\Phi_2 = \arctan \infty = 90^\circ$$

Since the phase angle is given by an inverse-tangent function, the phase angle is skew symmetric about the inflexion point at 45° .

The asymptotes, the phase angle and the exact log-magnitude curve are shown in Figure 6.6.

An advantage of the Bode diagram is that for the reciprocal factors, for example, the factor $Tj\omega + 1$, the log-magnitude and the phase angle curves need only be changed in sign. Since

$$20 \log \left| \frac{1}{T\omega + 1} \right| = -20 \log |T\omega + 1|$$

and

$$\angle \left(\frac{1}{T\omega + 1} \right) = -\angle(Tj\omega + 1)$$

the corner frequency is the same for both cases. The slope of the high-frequency curve is -20 dB/dec and the phase angle varies from 0 to -90° as the frequency is increased from zero to infinity. The log-magnitude, together with the asymptotes and the phase angle curve for the factor $\frac{1}{Tj\omega + 1}$ are shown in Figure 6.7.

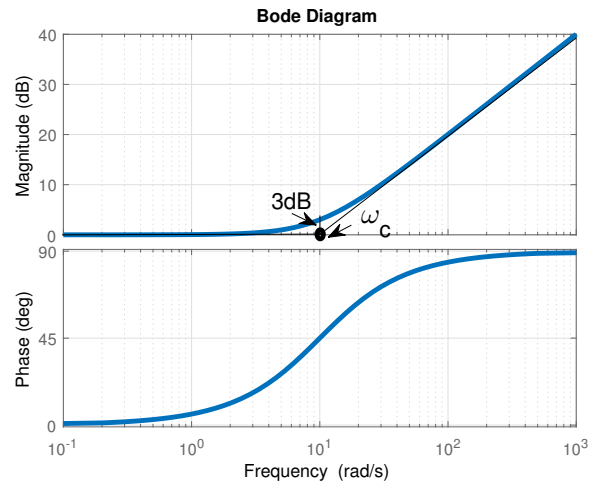
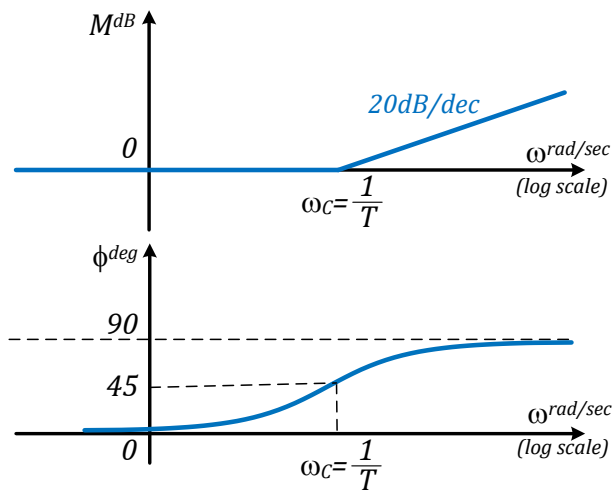


Figure 6.6: Logarithmic plot for first-order factors

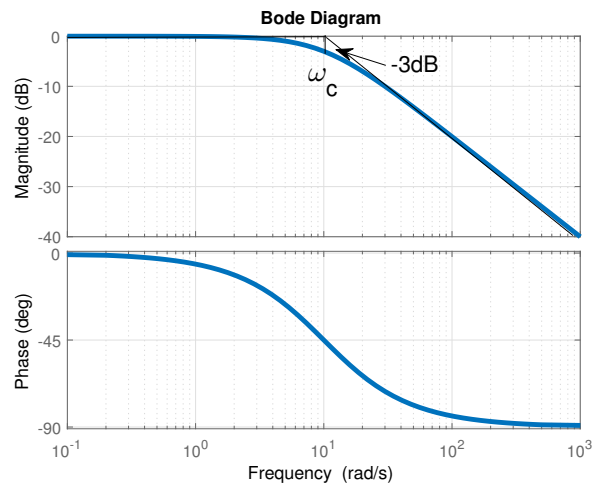
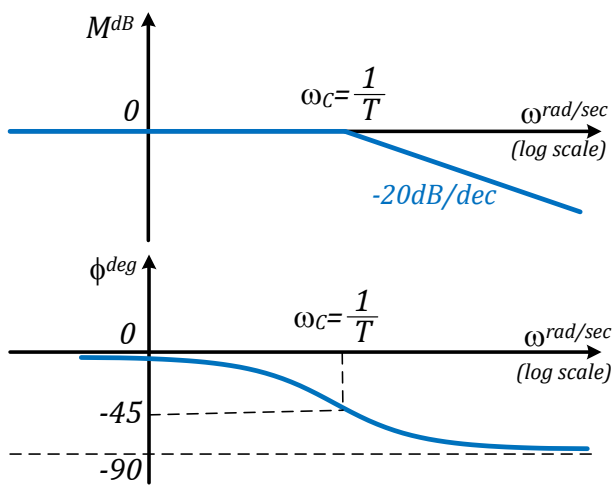


Figure 6.7: Logarithmic plot for first-order factors

3. Quadratic factors $[(\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1)]^{\pm 1}$ Control systems often posses quadratic factors of the form:

$$\frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1}$$

If $\zeta > 1$ this quadratic factor can be expressed as a product of two first-order ones with real poles. If $0 < \zeta < 1$, this quadratic factor has complex-conjugate poles. The asymptotic frequency-response curve may be obtained as follows: Since:

$$M_3^{dB} = 20 \log_{10} \left| \frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1} \right| = -20 \log_{10} \sqrt{(1 - \frac{\omega^2}{\omega_n^2})^2 + (2\zeta \frac{\omega}{\omega_n})^2}$$

for low frequencies such that $\omega \ll \omega_n$, the log-magnitude becomes:

$$M_3^{dB}|_{\omega \ll} = -20 \log_{10} 1 = 0dB$$

The low-frequency asymptote is thus a horizontal line at $0dB$. For high frequencies such that $\omega \gg \omega_n$, the log-magnitude becomes:

$$M_3^{dB}|_{\omega \gg} = -20 \log \frac{\omega^2}{\omega_n^2} = -40 \log_{10} \frac{\omega}{\omega_n} = -40 \log_{10} \omega - 40 \log_{10} \omega_n \text{ dB}$$

The equation for high-frequency asymptote is a straight line having the slope -40 dB/dec since:

$$\frac{dM_3^{dB}}{d\omega^{dec}} = -40 \text{ dB/dec}$$

The high-frequency asymptote intersects the low-frequency asymptote when:

$$M_3^{dB}|_{\omega \gg} = M_3^{dB}|_{\omega \ll}$$

or

$$-40 \log \omega_c - 40 \log \omega_n = 0$$

The corner (or break) frequency ω_c is:

$$\omega_c = \omega_n$$

The two asymptotes just derived are independent of the value of ζ . Near the corner frequency ω_c , a resonant peak occurs as may be expected. The damping ration ζ determines the magnitude of this peak.

The maximum error caused by the use of asymptotes occurs at the corner frequency and is calculated by replacing the expression of the corner frequency $\omega = \omega_c = \omega_n$ in the exact relation for log-magnitude:

$$\begin{aligned} M_3^{dB}|_{\omega=\omega_c} &= -20 \log_{10} \sqrt{(1 - \frac{\omega_n^2}{\omega_n^2})^2 + (2\zeta \frac{\omega_n}{\omega_n})^2} \\ &= -20 \log_{10}(2\zeta) = -20 \log_{10} 2 - 20 \log_{10} \zeta \cong -6^{dB} - \zeta^{dB} \end{aligned}$$

The magnitude of the error depends on the value of ζ . It is large for small values of ζ . Figure 6.8 shows the exact log-magnitude curves together with the asymptotes for several values of ζ .

The phase angle of the quadratic factor $\frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1}$ is:

$$\Phi_3 = \angle \left(\frac{1}{(\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1)} \right) = -\arctan \frac{2\zeta \frac{\omega}{\omega_n}}{1 - (\frac{\omega}{\omega_n})^2}$$

At $\omega = 0$, the phase angle equals 0.

At the corner frequency $\omega = \omega_c = \omega_n$, the phase angle is -90° regardless of ζ since:

$$\Phi_3 = -\arctan \frac{2\zeta}{0} = -\arctan \infty = -90^\circ$$

At $\omega \rightarrow \infty$, the phase angle becomes -180° . The phase angle is skew symmetric about the inflexion point, where $\Phi_3 = -90^\circ$. Figure 6.8 shows the exact phase angle for different values of ζ .

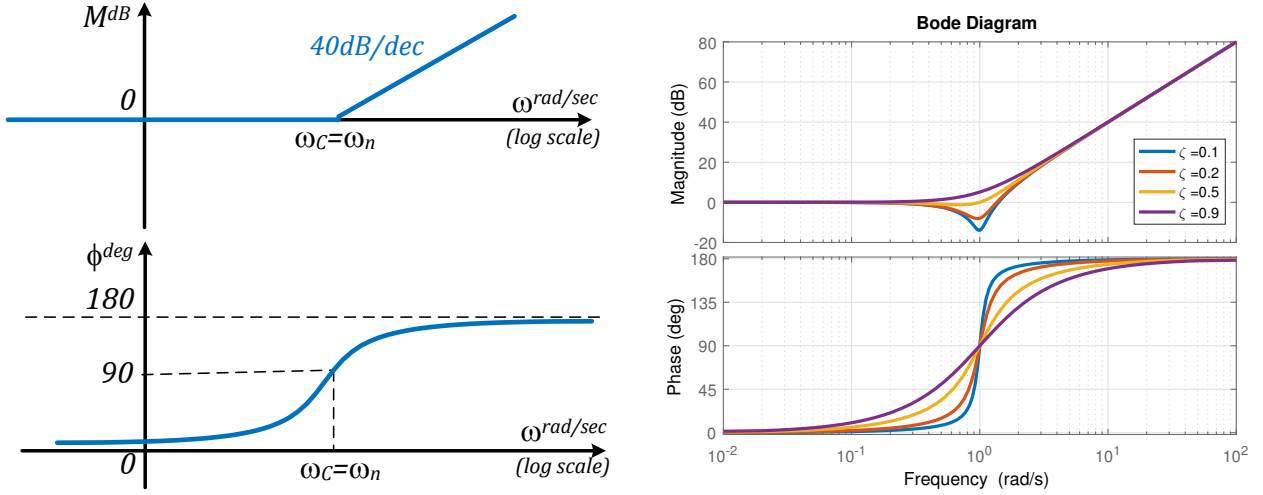


Figure 6.8: Logarithmic plot for second-order factors

The frequency response curves for the factor

$$\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1$$

can be obtained by reversing the sign of the log-magnitude and the phase angle of the factor $\frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1}$. The frequency response curves are presented in Figure 6.9.

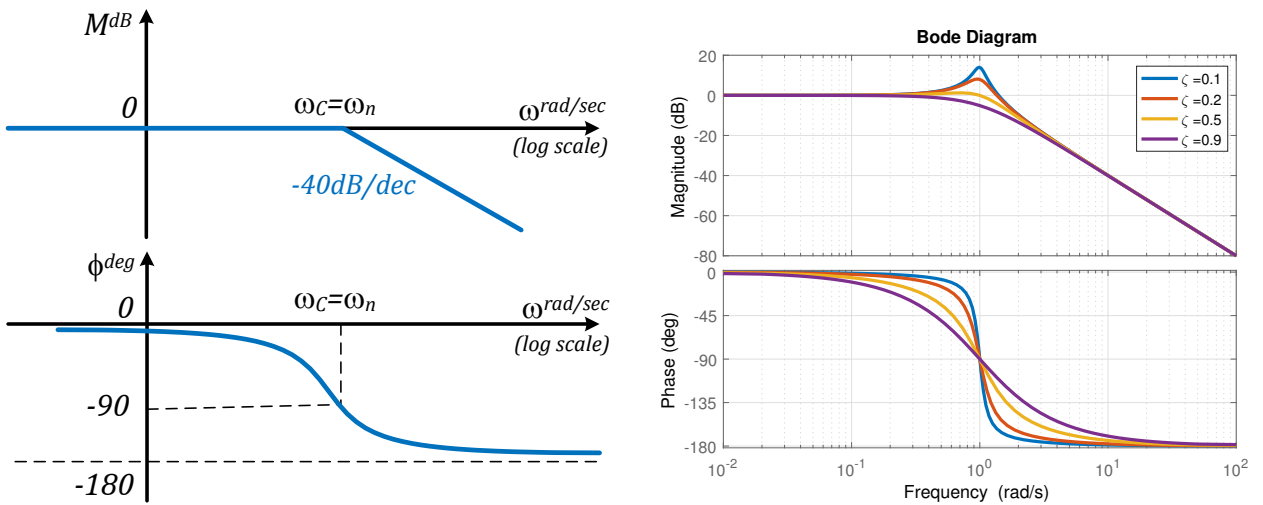


Figure 6.9: Logarithmic plot for second-order factors

Example 6.2.1 Draw the Bode diagram for the following transfer function:

$$G(s) = \frac{10^3(s + 10)}{s(s + 1)(s^2 + 10s + 100)}$$

First write the transfer function in the general form shown before, emphasizing the time constants, natural frequencies, damping factors and the gain:

$$G(s) = \frac{10^2(\frac{1}{10}s + 1)}{s(s + 1)(\frac{1}{100}s^2 + \frac{1}{10}s + 1)}$$

Replacing s with $j\omega$, this function is composed of the following factors:

$$\begin{aligned} G_1(j\omega) &= \frac{10^2}{j\omega} = \frac{k}{(j\omega)^1}; \\ G_2(j\omega) &= \frac{1}{10}(j\omega) + 1 = T_1(j\omega) + 1; \\ G_3(j\omega) &= \frac{1}{j\omega + 1} = \frac{1}{T_2j\omega + 1}; \\ G_4(j\omega) &= \frac{1}{\frac{1}{100}(j\omega)^2 + \frac{1}{10}(j\omega) + 1} = \frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1} \end{aligned} \quad (6.1)$$

The Bode diagram is plotted for every one of these factors and then the curves are added graphically. For simplicity reasons, only the asymptotes will be plotted for the log-magnitude curves. The gain k , time constants T_1 and T_2 , the damping factor ζ and the natural frequency ω_n , can be identified from equations (6.1), and we obtain:

$$k = 10^2; \quad T_1 = \frac{1}{10}; \quad T_2 = 1; \quad \omega_n = 10; \quad \zeta = 0.5.$$

The log-magnitude curve of $G_1(j\omega)$ is a straight line with the slope -20 dB/dec which crosses the vertical line for $\omega = 10^0 = 1$ at $k^{dB} = 20\log 10^2 = 40$ dB. The phase angle corresponding to this factor is a constant line at -90° . The plots are shown in Figure 6.10.

The magnitude curve of the first order factor at the numerator $G_2(j\omega)$ has two asymptotes: one at 0 dB for low frequencies and the other is a line with the slope 20 dB/dec. Both meet at the corner frequency

$$\omega_{c1} = \frac{1}{T_1} = 10 \text{ rad/sec}$$

The phase angle is an arctangent function which approaches the 0 dB line at low frequencies and 90° at very high frequencies with an inflexion point at $(\omega_{c1}, 45^\circ)$.

For the first-order factor at the denominator $G_3(j\omega)$ we obtain in a similar way the corner frequency

$$\omega_{c2} = \frac{1}{T_2} = 10^0 = 1 \text{ rad/sec}$$

and the asymptotes (the one at high frequencies will have a slope of 20 dB/dec).

The phase angle is an arctangent function which approaches the 0 dB line at low frequencies and -90° at very high frequencies with an inflexion point at $(\omega_{c2}, -45^\circ)$.

The log-magnitude curve of the second order factor at the denominator $G_4(j\omega)$ has a corner frequency:

$$\omega_{c3} = \omega_n = 10 \text{ rad/sec}$$

and two asymptotes: at 0 dB for low frequencies and one with a slope of -40 dB/dec at high frequencies.

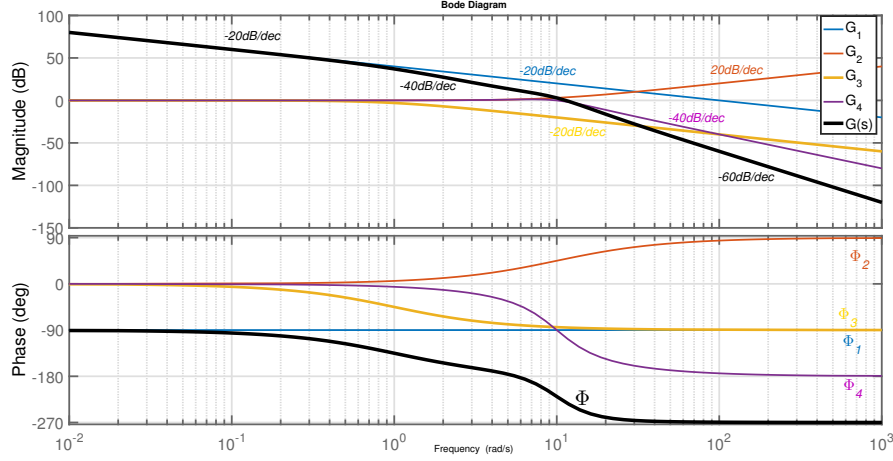


Figure 6.10: Logarithmic plot for Example 1

The phase angle approaches 0 dB at low frequencies and -180° at high frequencies with an inflexion point at $(\omega_{c3}, -90^\circ)$.

Adding the asymptotes. The total asymptotic magnitude can be plotted by adding algebraically the asymptotes due to each factor, as shown by the solid line in Figure 6.10.

Below $\omega = 10^0$ all the asymptotes approach the 0 dB line except for the plot of $G_1(j\omega)$ where the plot has a slope of -20 dB/dec. At the first corner frequency $\omega_{c2} = 10^0$, the slope changes to -40 dB/dec because of asymptote at high frequency of $G_3(j\omega)$ (which has a slope of -20 dB/dec) was added. The final plot changes again the slope at $\omega = 10^0$, where another two asymptotes at high frequencies add their slope to the resultant (-40 dB/dec + 20 dB/dec). Thus, for frequencies higher than $\omega = 10$ the final plot will have a slope of -60 dB/dec.

For plotting the complete phase-angle curve, the phase angle curves for all factors have to be sketched. The algebraic sum of these curves provides the complete phase-angle curve as shown in Figure 6.10.

Reading Bode Diagrams. If a sinusoidal signal $X(j\omega)$ with frequency ω is applied as the input to an open-loop system with the sinusoidal transfer function $G(j\omega)$ the output signal, $Y(j\omega)$ will have the same frequency. The amplitude of the output can be obtained by reading from a Bode diagram the magnitude of the transfer function corresponding to frequency ω and we obtain:

$$|Y(j\omega)| = |G(j\omega)| \cdot |X(j\omega)|$$

If $|G(j\omega)| > 1$ (or $M^{dB} = |G(j\omega)|^{dB} > 0$), the output will be amplified. If $|G(j\omega)| < 1$ (or $M^{dB} = |G(j\omega)|^{dB} < 0$), the output is attenuated.

The phase shift of the output signal with respect to in input is the phase angle of the transfer function represented in a Bode diagram. When $\Phi > 0$ the output is shifted with a positive angle comparing to the input and the system has phase lead. If $\Phi < 0$ the output is shifted with a negative angle with respect to the input (phase lag).

The log-magnitude curve shown in Figure 6.10 is positive for frequencies lower than $\omega = 10 \text{ rad/sec}$ and negative for the high frequencies. This means that for all low frequencies the system will amplify the input signal and will attenuate the signals with high frequencies. The system in this case is a low-pass filter.

A system which amplifies the high frequencies and attenuate the low ones is a high-pass filter.

6.3 Stability in the Frequency Domain

6.3.1 Bode plots and closed-loop system stability

It is important to determine whether a system is stable and if so, than the degree of stability is important to determine. We can use the frequency response of a transfer function around a feedback loop to provide answers to our questions about system's relative stability.

A frequency domain stability criterion relates the stability of a closed-loop system to the open-loop frequency response and open-loop pole location.

Our problem is the determination of stability from frequency data plots, for the closed-loop system.

Consider the closed-loop control system shown in Figure 6.11.

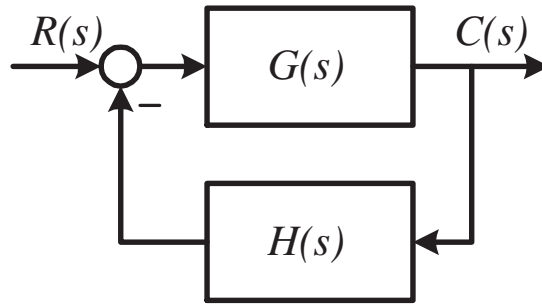


Figure 6.11: Closed-loop system

The zeros of the characteristic equation $1 + G(s)H(s) = 0$ are the poles of the closed-loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

Thus by examining the open loop transfer function, $G(s)H(s)$ (denoted here $GH(s)$), we can determine the stability (or otherwise) of the closed-loop system.

A point on the imaginary axis $s = j\omega$ will be a solution of the characteristic equation (i.e. the system is critically stable) if $|GH(j\omega)| = 1$ and $\angle GH(j\omega) = \pm 180^\circ$.

Since we have access to $|GH(j\omega)|$ and $\angle GH(j\omega)$ from a Bode plot, we should be able to determine the imaginary axis crossings by finding the frequencies ω (if any) on the plot that satisfy the conditions:

$$|GH(j\omega)| = 1 \quad \text{and} \quad \angle GH(j\omega) = \pm 180^\circ$$

We shall use the following terminology:

- **Gain crossover frequency:** This is the frequency ω_g such that $|GH(j\omega_g)| = 1$ (or equivalently, $20 \log_{10} |GH(j\omega_g)| = M^{dB}(\omega_g) = 0$).
- **Phase crossover frequency:** This is the frequency ω_p such that $\angle GH(j\omega_p) = \pm 180^\circ$.

Example 1. Consider a closed-loop system having the open-loop transfer function:

$$GH(s) = \frac{K}{s(s+1)(\frac{1}{100}s+1)}$$

The Bode plot is given in Figure 6.12 for $K = 1$.

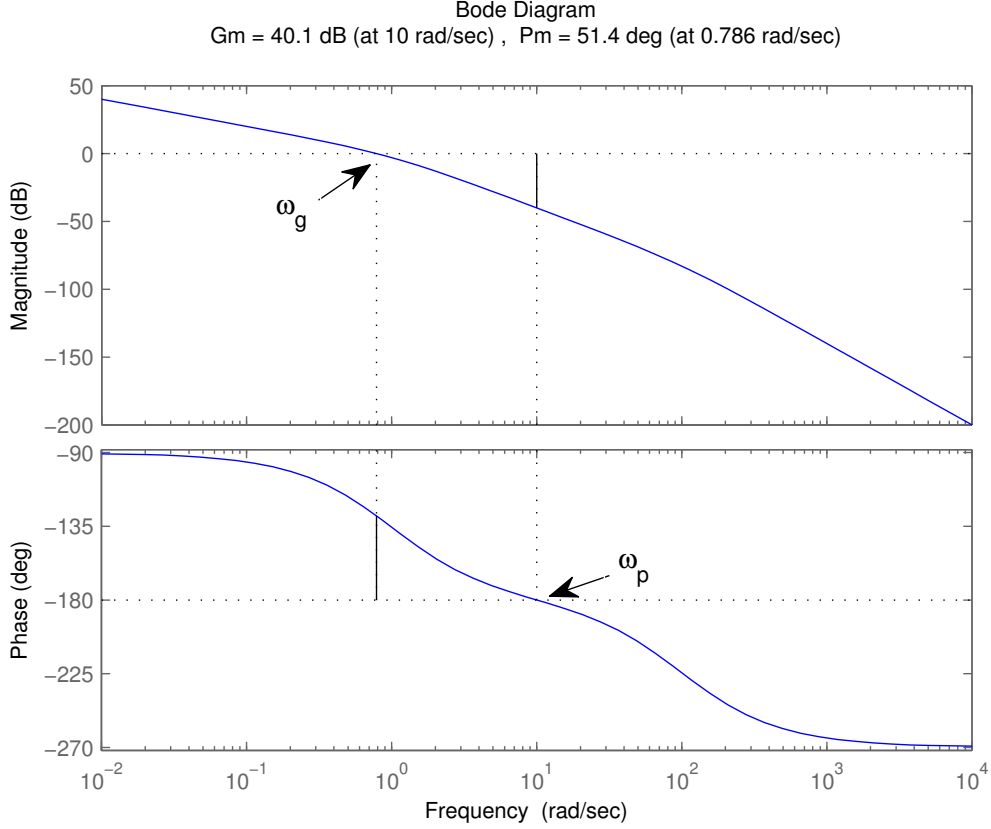


Figure 6.12: Example 1. Bode plot

In this example, we have $\omega_g \approx 1[\text{rad/sec}]$ (or $\omega_g^{\text{dec}} = 0 \text{ dec}$). The phase at ω_g is approximately -135° , and so the feedback configuration with $K=1$ does not have any closed-loop poles on the imaginary axis. We may determine whether there is another value of K for which the closed-loop system will have poles on the imaginary axis (and this cross the boundary between stability to instability). Note that:

$$20 \cdot \log_{10} K \cdot GH(j\omega) = 20 \cdot \log_{10} K + 20 \cdot \log_{10} GH(j\omega)$$

and

$$\angle K \cdot GH(j\omega) = \angle GH(j\omega), \text{ for } K > 0$$

Thus, K has no effect on the phase and it affects the magnitude plot by shifting it up and down by K^{dB} . For example, when $K = 10$, the entire magnitude gets shifted up by $20 \log_{10} 10 = 20 \text{ dB}$. Examining the magnitude plot, we see that it has to be shifted up by approximately 40 dB in order to set $\omega_g = \omega_p$, which can be accomplished by setting $K = 100$ (or $K^{\text{dB}} = 40$). One can easily see from the Bode plot that this is the only positive value of K for which this will happen. Figure 6.13 shows the changes in the magnitude plot for various values of K .

Relative stability

In designing a control system, we require that the closed-loop system be stable. Furthermore, it is necessary that the system have adequate relative stability. In this section we shall show that the Bode plot indicates not only whether or not a system is stable but also the degree of stability of a stable system.

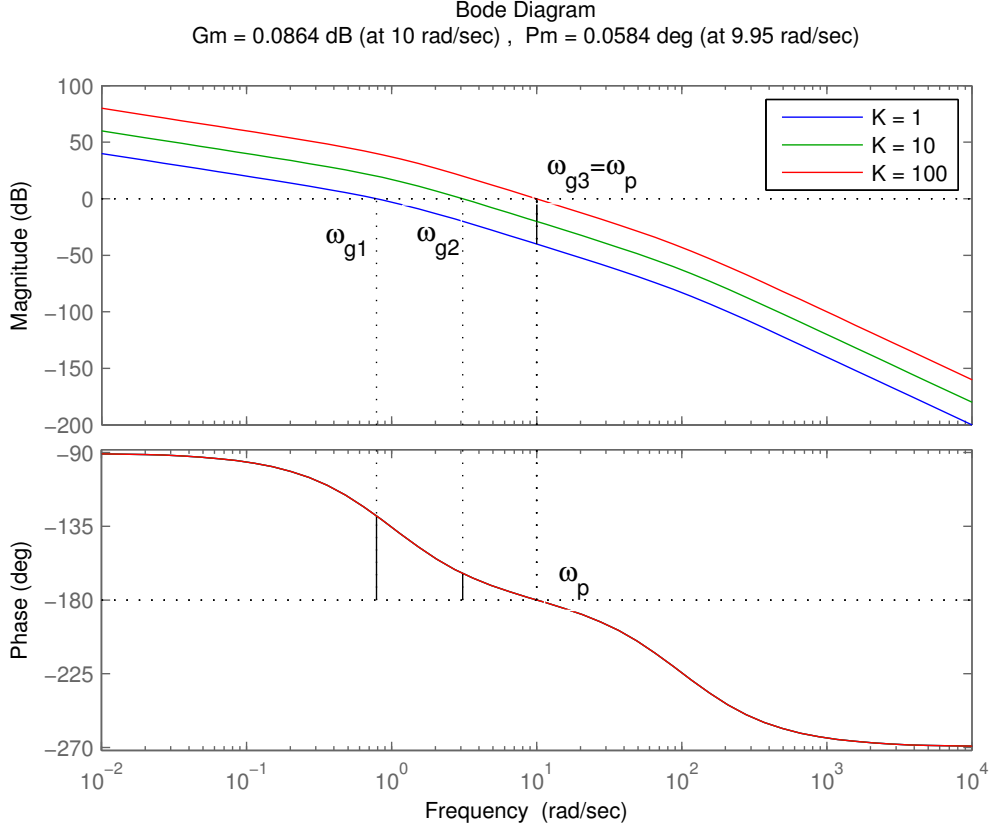


Figure 6.13: Example 1. Bode plot for various values of K

6.3.2 Phase and Gain Margin

Assuming that the closed-loop system is stable, we can determine how far from instability is the system. There are two metrics to evaluate this:

- **Gain margin:** is the amount by which K can be multiplied before $|KGH(j\omega_p)| = 1$, or $20\log|KGH(j\omega_p)| = M^{dB}(\omega_p) = 0\text{dB}$ (i.e. the gain crossover frequency and phase crossover frequencies coincide). In other words, the *gain margin* is the reciprocal of the gain $|GH(j\omega)|$ at the frequency at which the phase angle reaches -180° . The gain margin indicates how much the gain can be increased before the system becomes unstable.
- **Phase margin:** is the amount by which the phase at ω_g exceeds -180° .

More specifically, they are defined by:

- **Gain margin K_g :**

$$K_g = \frac{1}{|GH(j\omega_p)|}, \text{ for } \angle GH(j\omega_p) = -180^\circ$$

or, in log scale:

$$K_g^{dB} = -M^{dB}(\omega_p)$$

- **Phase margin, γ :**

$$\gamma = 180^\circ + \angle GH(j\omega_g), \text{ for } M(\omega_g) = |GH(j\omega_g)| = 1, \text{ or } M^{dB}(\omega_g) = 0$$

6.3.3 Stability analysis

- A closed-loop system will be stable if the gain margin and phase margin are positive.
- Negative margins indicate instability.
- For satisfactory performance, the phase margin should be between 30° and 60° and gain margin should be greater than $6dB$.
- Either the gain margin or the phase margin alone does not give a sufficient indication of the relative stability. Both should be given in order to determine the relative stability.
- For first order and second order system, gain margin is always infinity.

6.3.3.1 Example1. Stable closed-loop system

Consider a **closed-loop** control system having the **open-loop** transfer function

$$GH(s) = \frac{1}{s(s+1)^2}$$

The closed-loop system stability and relative stability will be assessed using the Bode plot for the open loop system, shown in Figure 6.14.

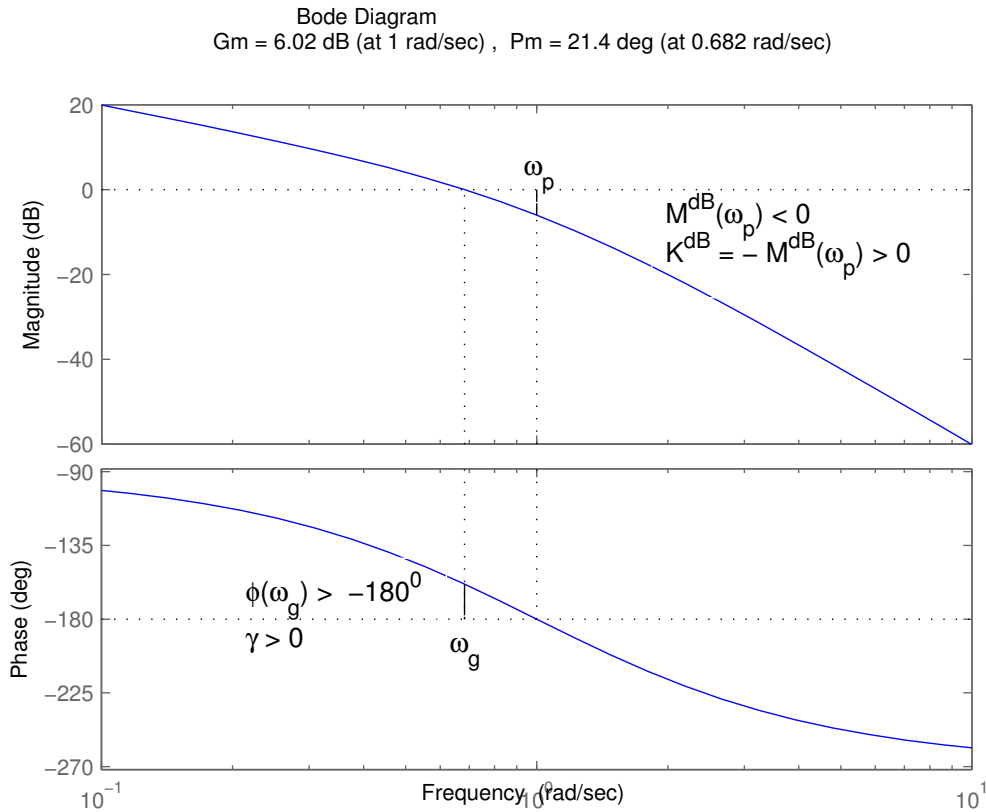


Figure 6.14: Phase margin and gain margin for a stable closed-loop system

From the plot, the gain margin is evaluated at the frequency ω_p at which the phase crosses -180° line. The magnitude (in dB) is negative, thus the gain margin will be positive: $K^{dB} = -M^{dB}(\omega_p)$.

The phase margin is read at the frequency ω_g , when $M^{dB}(\omega_g) = 0$. It is the difference between -180° and the phase angle of the open-loop system measured at this frequency. Note that in the relation:

$$\gamma = 180^\circ + \angle GH(j\omega_g),$$

the angle $\angle GH(j\omega_g)$ is negative, thus the phase angle $\gamma > 0$.

For this example, the gain and phase margin are:

$$K^{dB} = 6.02dB, \quad \gamma = 21.4^\circ$$

and the closed-loop system is stable.

6.3.3.2 Example2. Unstable closed-loop system

Consider a **closed-loop** control system having the **open-loop** transfer function

$$GH(s) = \frac{100}{s(s+1)^2}$$

The closed-loop system stability and relative stability will be assessed using the Bode plot for the open loop system, shown in Figure 6.15.

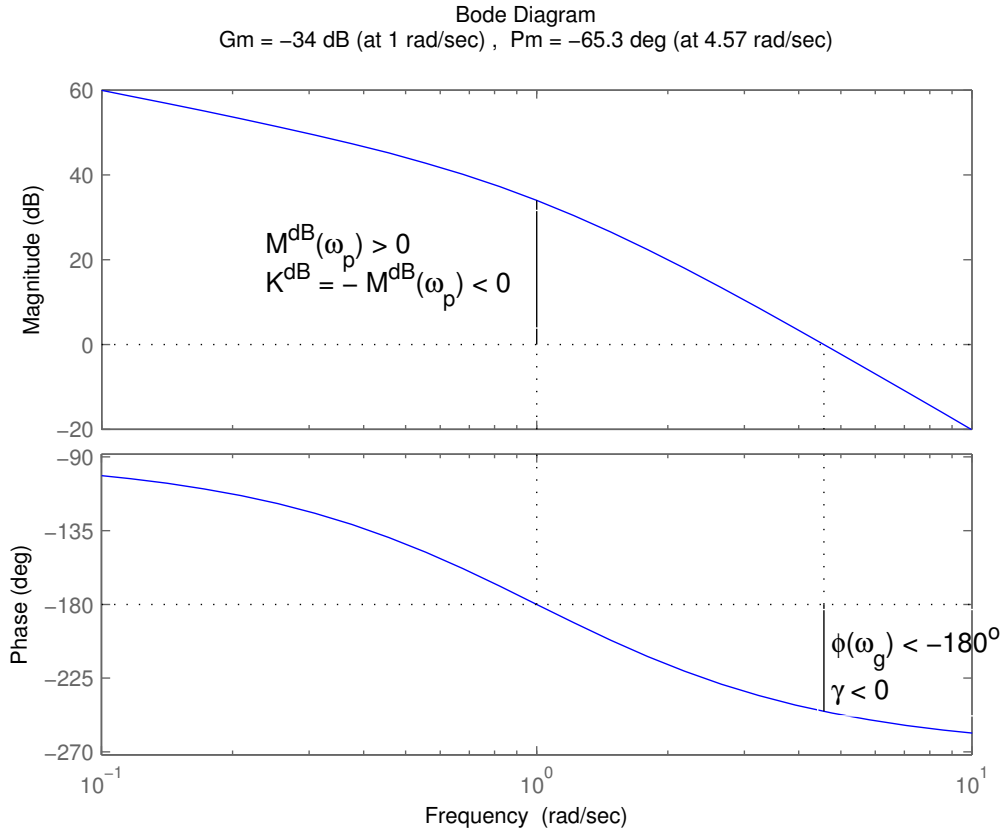


Figure 6.15: Phase margin and gain margin for an unstable closed-loop system

In this case, the gain margin (expressed in dB) is negative because the magnitude is positive at the phase crossover frequency. The phase margin is negative too, because the phase angle of the open-loop system measured at the gain crossover frequency is less than 180° .

For this example, the gain and phase margin are:

$$K^{dB} = -34dBdB, \quad \gamma = -65.3^\circ$$

and the closed-loop system is unstable.

6.3.3.3 Example3

Consider a **closed-loop** control system having the **open-loop** transfer function

$$GH(s) = \frac{10k(s+1)}{s^2(s+10)}$$

The closed-loop system stability and relative stability will be assessed using the Bode plot for the open loop system, shown in Figure 6.16, for $k = 1$.

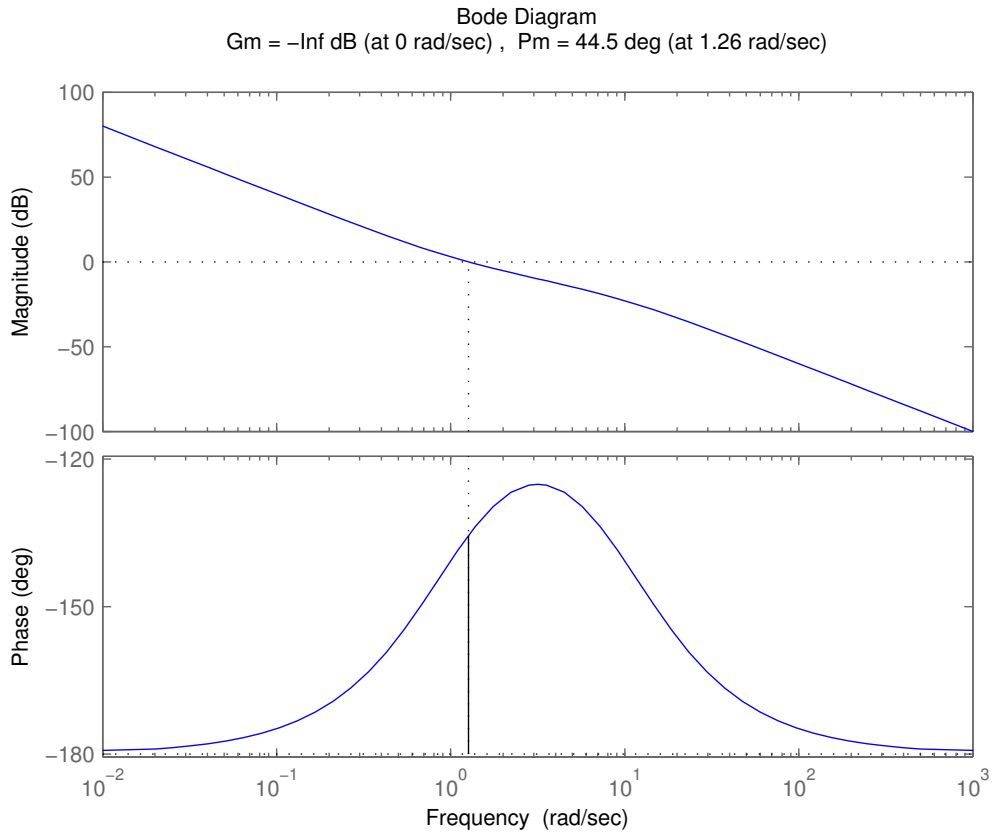


Figure 6.16: Infinite gain margin

For this example, the gain and phase margin are:

$$K^{dB} = Inf dB, \quad \gamma = 44.5^\circ$$

and the closed-loop system is stable.

The gain margin cannot be evaluated from the Bode plot because the phase angle plot does not cross the -180° line. In this case, the closed-loop system is stable whatever the value of the gain K . This can be verified from a root locus plot or by using the Routh-Hurwitz criterion.

Bibliography

Dorf, R. C. and Bishop, R. H. (2008). *Modern Control Systems*. Pearson Prentice Hall, 11th edition.